

On the correlation functions of the characteristic polynomials of the hermitian sample covariance ensemble

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Abstract

We consider asymptotic behavior of the correlation functions of the characteristic polynomials of the hermitian sample covariance matrices $H_n = n^{-1}A_{m,n}^*A_{m,n}$, where $A_{m,n}$ is a $m \times n$ complex matrix with independent and identically distributed entries $\Re a_{\alpha j}$ and $\Im a_{\alpha j}$. We show that for the correlation function of any even order the asymptotic behavior in the bulk and at the edge of the spectrum coincides with those for the Gaussian Unitary Ensemble up to a factor, depending only on the fourth moment of the common probability law of entries $\Re a_{\alpha j}$, $\Im a_{\alpha j}$, i.e. the higher moments do not contribute to the above limit.

1 Introduction

Characteristic polynomials of random matrices have been actively studied in the last years. The interest was initially stimulated by the similarity between the asymptotic behavior of the moments of characteristic polynomials of a random matrix from the Circular Unitary Ensemble and the moments of the Riemann ζ -function along its critical line (see [13]). But with the emerging connections to the quantum chaos, integrable systems, combinatorics, representation theory and others, it has become apparent that the characteristic polynomials of random matrices are also of independent interest. This motivates the studies of the moments of characteristic polynomials for other random matrix ensembles (see e.g. [11], [15], [4], [8], [1], [19], [20], [5], [10], [21]).

In this paper we consider the hermitian sample covariance ensembles with symmetric entries distributions, i.e. $n \times n$ random matrices of the form

$$H_n = n^{-1}A_{m,n}^*A_{m,n}, \quad (1.1)$$

where $A_{m,n}$ is an $m \times n$ complex matrix with independent and identically distributed entries $\Re a_{\alpha j}$ and $\Im a_{\alpha j}$ such that

$$\begin{aligned} \mathbf{E}\{a_{\alpha j}\} &= \mathbf{E}\{(a_{\alpha j})^2\} = 0, & \mathbf{E}\{|a_{\alpha j}|^2\} &= 1, & \alpha &= 1, \dots, m, j = 1, \dots, n, \\ \mathbf{E}\{\Re^{2l+1} a_{\alpha j}\} &= \mathbf{E}\{\Im^{2l+1} a_{\alpha j}\} = 0, & l &\in \mathbb{N}. \end{aligned} \quad (1.2)$$

We assume that m belongs to a sequence $\{m_n\}_{n=1}^\infty$ such that

$$c_{m,n} := \frac{m_n}{n} \rightarrow c \geq 1, \quad n \rightarrow \infty. \quad (1.3)$$

We below denote this limit as " $\lim_{m,n \rightarrow \infty} \dots$ ".

Let $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ be the eigenvalues of H_n . Define their Normalized Counting Measure (NCM) as

$$N_n(\Delta) = \#\{\lambda_j^{(n)} \in \Delta, j = 1, \dots, n\}/n, \quad N_n(\mathbb{R}) = 1, \quad (1.4)$$

where Δ is an arbitrary interval of the real axis. The behavior of N_n , as $n \rightarrow \infty$, is studied well enough. In particular, it was shown in [17] that N_n converges weakly in probability to a non-random measure N which is called the limiting NCM of the ensemble. The measure N is absolutely continuous and its density ρ is given by the well-known Marchenko-Pastur law:

$$\rho(\lambda) = \begin{cases} \frac{1}{2\pi\lambda} \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}, & \lambda \in \sigma, \\ 0, & \lambda \notin \sigma, \end{cases} \quad (1.5)$$

where

$$\lambda_{\pm} = (1 \pm \sqrt{c})^2, \quad \sigma = ((1 - \sqrt{c})^2, (1 + \sqrt{c})^2). \quad (1.6)$$

The mixed moments (or the correlation functions) of characteristic polynomials are

$$F_{2k}(\Lambda) = \int_{\mathcal{H}_n^+} \prod_{j=1}^{2k} \det(\lambda_j - H_n) P_n(dH_n), \quad (1.7)$$

where \mathcal{H}_n^+ is the space of positive definite hermitian $n \times n$ matrices, $P_n(dH_n)$ is a probability law of the $n \times n$ random matrix H_n , and $\Lambda = \{\lambda_j\}_{j=1}^{2k}$ are real or complex parameters that may depend on n .

We are interested in the asymptotic behavior of (1.7) for matrices (1.1) as $m, n \rightarrow \infty$ and for $j = 1, \dots, 2k$

$$\lambda_j = \begin{cases} \lambda_0 + \xi_j/n\rho(\lambda_0), & \lambda_0 \in \sigma, \\ \lambda_0 + \xi_j/(n\gamma_{\pm})^{2/3}, & \lambda_0 = \lambda_{\pm}, \end{cases} \quad (1.8)$$

where λ_{\pm} and σ are defined in (1.6),

$$\gamma_{\pm} = \frac{c^{1/4}}{(1 \pm \sqrt{c})^2}, \quad (1.9)$$

ρ is defined in (1.5), and $\widehat{\xi} = \{\xi_j\}_{j=1}^{2k}$ are real parameters varying in $[-M, M] \subset \mathbb{R}$.

In the case of hermitian matrix models the asymptotic behavior of (1.7) was obtained by using the method of orthogonal polynomials (see [3, 19]). Unfortunately, the method of orthogonal polynomials can not be applied to the general case of the hermitian sample covariance ensembles (1.1) – (1.2). In the paper [21] the method based on the Grassmann integration was developed to study the asymptotic behavior of the correlation functions of any any even number of the characteristic polynomials of the hermitian Wigner ensemble. Here we apply this method to the hermitian sample covariance ensembles (1.1) – (1.2).

In [12] Kusters use the exponential generating function to study the second moment, i.e. the case $k = 1$ in (1.7). It was shown that for $\lambda_0 \in \sigma$

$$\begin{aligned} \frac{1}{n\rho(\lambda_0)} F_2(\lambda_0 + \xi_1/(n\rho(\lambda_0)), \lambda_0 + \xi_2/(n\rho(\lambda_0))) &= 2\pi\lambda_0^{n-m} c_{m,n}^{m+1/2} \\ &\times e^{-n-m} \exp\{n\lambda_0 + \alpha(\lambda_0)(\xi_1 + \xi_2) + 2\kappa_4\} \frac{\sin(\pi(\xi_1 - \xi_2))}{\pi(\xi_1 - \xi_2)} (1 + o(1)), \end{aligned}$$

where

$$\alpha(\lambda_0) = \begin{cases} \frac{\lambda_0 - c + 1}{2\lambda_0\rho(\lambda_0)}, & \lambda_0 \in \sigma, \\ (1 \pm \sqrt{c})^{-1}\gamma_{\pm}^{-2/3}, & \lambda_0 = \lambda_{\pm}, \end{cases} \quad \kappa_4 = \mu_4 - 3/4, \quad (1.10)$$

γ_{\pm} is defined in (1.9), and μ_4 is the fourth moment of the probability law of $\Im W_{jk}$, $\Re W_{jk}$. In [12] for the case $c > 1$, $m = cn + o(n^{1/3})$, $k = 1$ the asymptotic behavior at the edge of the spectrum (i.e. for $\lambda_0 = \lambda_{\pm}$) was also obtained:

$$\begin{aligned} \frac{1}{(n\gamma_{\pm})^{2/3}} F_2(\lambda_0 + \xi_1/(n\gamma_{\pm})^{2/3}, \lambda_0 + \xi_2/(n\gamma_{\pm})^{2/3}) &= 2\pi(1 \pm \sqrt{c})^{2(n-m)} c^{m+1/2} \\ &\times e^{2n\sqrt{c}} \exp\{n^{1/3}\alpha(\lambda_{\pm})(\xi_1 + \xi_2) + 2\kappa_4\} A(\xi_1, \xi_2) (1 + o(1)) \end{aligned}$$

with

$$A(x, y) = \frac{\text{Ai}'(x)\text{Ai}(y) - \text{Ai}(x)\text{Ai}'(y)}{x - y}, \quad (1.11)$$

where $\text{Ai}(x)$ is the Airy function

$$\text{Ai}(x) = \frac{1}{2\pi} \int_S e^{is^3/3 + isx} ds, \quad (1.12)$$

$$S = \{z \in \mathbb{C} \mid \arg z = \pi/6 \text{ or } \arg z = 5\pi/6\}.$$

In this paper we consider the general case $k \geq 1$ of (1.7) for the random matrices (1.1). Define

$$D^{(n)}(\xi, \lambda_0) = \begin{cases} (n\rho(\lambda_0))^{-1} F_2\left(\lambda_0 + \xi/(n\rho(\lambda_0)), \lambda_0 + \xi/(n\rho(\lambda_0))\right), & \lambda_0 \in \sigma, \\ (n\gamma_{\pm})^{-2/3} F_2\left(\lambda_0 + \xi/(n\gamma_{\pm})^{2/3}, \lambda_0 + \xi/(n\gamma_{\pm})^{2/3}\right), & \lambda_0 = \lambda_{\pm}, \end{cases} \quad (1.13)$$

and denote

$$D_{2k}(\lambda_0) = \prod_{l=1}^{2k} \sqrt{D^{(n)}(\xi_l, \lambda_0)}. \quad (1.14)$$

The main results of the paper are the following two theorems:

Theorem 1. *Let the entries $\Im a_{\alpha j}$, $\Re a_{\alpha j}$ of the matrices (1.1) have a symmetric probability distribution with finite first $4k$ moments. Then we have for $k \geq 1$*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{(n\rho(\lambda_0))^{k^2} D_{2k}(\lambda_0)} F_{2k}\left(\Lambda_0 + \widehat{\xi}/(n\rho(\lambda_0))\right) \\ = \frac{c^{k(k-1)/2} \exp\{k(k-1)\kappa_4(c - \lambda_0 + 1)^2 c^{-1}\}}{\Delta(\xi_1, \dots, \xi_m) \Delta(\xi_{k+1}, \dots, \xi_{2k})} \det \left\{ \frac{\sin(\pi(\xi_i - \xi_{k+j}))}{\pi(\xi_i - \xi_{k+j})} \right\}_{i,j=1}^k, \end{aligned} \quad (1.15)$$

where F_{2k} and $\rho(\lambda)$ are defined in (1.7) and (1.5), $\Lambda_0 = (\lambda_0, \dots, \lambda_0) \in \mathbb{R}^{2k}$, $\lambda_0 \in \sigma$, $\widehat{\xi} = \{\xi_j\}_{j=1}^{2k}$, and κ_4 and σ are defined in (1.10) and (1.6).

Theorem 2. *Let the entries $\Im a_{\alpha j}$, $\Re a_{\alpha j}$ of the matrices (1.1) have a symmetric probability distribution with finite first $4k$ moments, $\lambda = \lambda_{\pm}$ and let m belong to a sequence $\{m_n\}_{n=1}^{\infty}$ such that*

$$m_n = cn + n^{1/3}\varepsilon_n, \quad c > 1, \quad (1.16)$$

where $\varepsilon_n \rightarrow 0$, $n \rightarrow \infty$. Then we have for $k \geq 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{(n\gamma_{\pm})^{2k^2/3} D_{2k}(\lambda_{\pm})} F_{2k} \left(\Lambda_0 + \widehat{\xi}/(n\gamma_{\pm})^{2/3} \right) \\ = \frac{c^{k(k-1)/2} \exp\{4k(k-1)\kappa_4\}}{\Delta(\xi_1, \dots, \xi_m) \Delta(\xi_{k+1}, \dots, \xi_{2k})} \det \left\{ A(\xi_j, \xi_{k+l}) \right\}_{i,j=1}^k, \end{aligned}$$

where F_{2k} and γ_{\pm} are defined in (1.7) and (1.9), $\Lambda_0 = (\lambda_{\pm}, \dots, \lambda_{\pm}) \in \mathbb{R}^{2k}$, $\widehat{\xi} = \{\xi_j\}_{j=1}^{2k}$, and κ_4 and λ_{\pm} are defined in (1.10) and (1.6).

The theorems show that the above limits for the mixed moments of the characteristic polynomials for random matrices (1.1) coincide with those for the Gaussian Unitary Ensemble up to a factor depending only on the fourth moment of the common probability law of the entries $a_{\alpha j}$, i.e., that the higher moments of the law do not contribute to the above limit. This is a manifestation of the universality, that can be compared with the universality of the local bulk regime for Wigner matrices (see [7] and references therein).

The paper is organized as follows. In Section 2 we obtain a convenient asymptotic integral representation for F_{2k} , using the integration over the Grassmann variables and the Harish Chandra/Itzykson-Zuber formula for integrals over the unitary group. The method is similar to that of [21]. In Section 3 and 4 we prove Theorem 1 and 2, applying the steepest descent method to the integral representation.

We denote by C, C_1 , etc. various n -independent quantities below, which can be different in different formulas.

2 The integral representation.

In this section we obtain the integral representation for the mixed moments F_{2k} (1.7) of the characteristic polynomials, i.e. we prove the following proposition

Proposition 1. *Let $\Lambda_{2k} = \Lambda_0 + \widehat{\xi}/(an)^{\alpha}$, where $\Lambda_0 = \text{diag}\{\lambda_0, \dots, \lambda_0\}$, $\widehat{\xi} = \text{diag}\{\xi_1, \dots, \xi_{2k}\}$, and*

$$a = \begin{cases} \rho(\lambda_0), & \lambda_0 \in \sigma, \\ \gamma_{\pm}, & \lambda_0 = \lambda_{\pm}, \end{cases} \quad (2.1)$$

$$\beta = \begin{cases} 1, & \lambda_0 \in \sigma, \\ 2/3, & \lambda_0 = \lambda_{\pm}, \end{cases} \quad (2.2)$$

where σ and λ_{\pm} are defined in (1.6), and let $F_{2k}(\Lambda_{2k})$ of (1.7) be the correlation function of the characteristic polynomials. Then we have for every k

$$D_{2k}^{-1}(\lambda_0)F_{2k}(\Lambda_{2k}) = \frac{n^{2k^2}(n^{\beta-1}a^{\beta})^{k(2k-1)}}{2^k\pi^k e^{2kn}D_{2k}(\lambda_0)} \oint_{\omega} \prod_{j=1}^{2k} dv_j e^{\sum_{j=1}^{2k} (n\lambda_0 v_j + n^{1-\beta} a^{-\beta} \xi_j v_j)} \frac{\Delta(V)}{\Delta(\widehat{\xi})} \quad (2.3)$$

$$\prod_{l=1}^{2k} \frac{(1-v_l)^m}{v_l^{n+2k}} \exp \left\{ 2c_{m,n} \kappa_4 S_2((I-V)\Lambda_0) \prod_{l=1}^{2k} \frac{v_l}{1-v_l} \right\} (1 + O(n^{-\beta})), \quad m, n \rightarrow \infty,$$

where $V = \text{diag}\{v_1, \dots, v_{2k}\}$,

$$S_2(A) = \frac{1}{2} \frac{d^2}{dx^2} \det(x - A) \Big|_{x=c}, \quad (2.4)$$

$D_{2k}(\lambda_0)$ is defined in (1.14) and ω is any closed contour encircling 0.

To this end we use the integration over the Grassmann variables. The integration was introduced by Berezin and widely used in the physics literature (see e.g. [2] and [6]). For the reader convenience we give a brief outline of the techniques here.

2.1 Grassmann integration

Let us consider two sets of formal variables $\{\psi_j\}_{j=1}^n, \{\bar{\psi}_j\}_{j=1}^n$, which satisfy the anticommutation conditions

$$\psi_j \psi_k + \psi_k \psi_j = \bar{\psi}_j \psi_k + \psi_k \bar{\psi}_j = \bar{\psi}_j \bar{\psi}_k + \bar{\psi}_k \bar{\psi}_j = 0, \quad j, k = 1, \dots, n. \quad (2.5)$$

These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\bar{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra \mathfrak{A} . Taking into account that $\psi_j^2 = 0$, we have that all elements of \mathfrak{A} are polynomials of $\{\psi_j\}_{j=1}^n$ and $\{\bar{\psi}_j\}_{j=1}^n$. We can also define functions of the Grassmann variables. Let χ be an element of \mathfrak{A} . For any analytical function f we mean by $f(\chi)$ the element of \mathfrak{A} obtained by substituting χ in the Taylor series of f . Since χ is a polynomial of $\{\psi_j\}_{j=1}^n, \{\bar{\psi}_j\}_{j=1}^n$, there exists such l that $\chi^l = 0$ and hence the series terminates after a finite number of terms and so $f(\chi) \in \mathfrak{A}$.

Note also that if χ is the sum of the products of even numbers of the Grassmann variables, then, according to the definition of the functions of the Grassmann variables, expanding $(z - \chi)^{-1}$ into the series we obtain for any analytic function f

$$\oint_{\Omega} \frac{f(z)}{z - \chi} \frac{dz}{2\pi i} = f(\chi), \quad (2.6)$$

where Ω is any closed contour encircling 0.

Following Berezin [2], we define the operation of integration with respect to the anti-commuting variables in a formal way:

$$\int d\psi_j = \int d\bar{\psi}_j = 0, \quad \int \psi_j d\psi_j = \int \bar{\psi}_j d\bar{\psi}_j = 1. \quad (2.7)$$

This definition can be extended on the general element of \mathfrak{A} by the linearity. A multiple integral is defined to be a repeated integral. The "differentials" $d\psi_j$ and $d\bar{\psi}_k$ anticommute with each other and with the variables ψ_j and $\bar{\psi}_k$.

Thus, if

$$f(\chi_1, \dots, \chi_k) = a_0 + \sum_{j_1=1}^k a_{j_1} \chi_{j_1} + \sum_{j_1 < j_2} a_{j_1 j_2} \chi_{j_1} \chi_{j_2} + \dots + a_{1,2,\dots,k} \chi_1 \dots \chi_k,$$

then

$$\int f(\chi_1, \dots, \chi_k) d\chi_k \dots d\chi_1 = a_{1,2,\dots,k}.$$

Let A be an ordinary hermitian matrix. The following Gaussian integral is well-known

$$\int \exp \left\{ - \sum_{j,k=1}^n A_{j,k} z_j \bar{z}_k \right\} \prod_{j=1}^n \frac{d\Re z_j d\Im z_j}{\pi} = \frac{1}{\det A}. \quad (2.8)$$

One of the important formulas of the Grassmann variables theory is the analog of this formula for the Grassmann algebra (see [2]):

$$\int \exp \left\{ \sum_{j,k=1}^n A_{j,k} \bar{\psi}_j \psi_k \right\} \prod_{j=1}^n d\bar{\psi}_j d\psi_j = \det A. \quad (2.9)$$

Besides, we have

$$\int \prod_{p=1}^q \bar{\psi}_{l_p} \psi_{s_p} \exp \left\{ \sum_{j,k=1}^n A_{j,k} \bar{\psi}_j \psi_k \right\} \prod_{j=1}^n d\bar{\psi}_j d\psi_j = \det A_{l_1,\dots,l_q;s_1,\dots,s_q}, \quad (2.10)$$

where $A_{l_1,\dots,l_q;s_1,\dots,s_q}$ is a $(n-q) \times (n-q)$ minor of the matrix A without lines l_1, \dots, l_q and columns s_1, \dots, s_q .

2.2 Asymptotic integral representation for F_2

In this subsection we obtain (2.3) for $k = 1$ by using the Grassmann integrals. This formula was obtained in [12] by using another method. We give here a detailed proof to show the basic ingredients of our techniques that will be elaborated in the next subsection to obtain the asymptotic integral representation of (1.7) for $k > 1$.

Using (2.9), we obtain from (1.7)

$$\begin{aligned} F_2(\Lambda_2) &= \mathbf{E} \left\{ \int e^{\sum_{p,q=1}^2 (\lambda_l - H)_{p,q} \bar{\psi}_{pr} \psi_{qr}} d\Psi_{2,n} \right\} \\ &= \mathbf{E} \left\{ \int \prod_{\alpha=1}^m e^{-\frac{1}{n} \sum_{r=1}^2 \left(\sum_{p=1}^n \bar{a}_{\alpha p} \bar{\psi}_{pr} \right) \left(\sum_{q=1}^n a_{\alpha q} \psi_{qr} \right)} e^{\sum_{s=1}^2 \lambda_s \sum_{p=1}^n \bar{\psi}_{ps} \psi_{ps}} d\Psi_{2,n} \right\} \\ &= \mathbf{E} \left\{ \int \prod_{\alpha=1}^m \prod_{r=1}^2 \left(1 - \frac{1}{n} \sum_{p,q=1}^n \bar{a}_{\alpha p} a_{\alpha q} \bar{\psi}_{pr} \psi_{qr} \right) e^{\sum_{s=1}^2 \lambda_s \sum_{p=1}^n \bar{\psi}_{ps} \psi_{ps}} d\Psi_{2,n} \right\}, \end{aligned} \quad (2.11)$$

since for any $\alpha = 1, \dots, m$ and any $r = 1, 2$ we have according to (2.5)

$$\left(\sum_{p=1}^n \bar{a}_{\alpha p} \bar{\psi}_{pr} \right)^2 = \left(\sum_{q=1}^n a_{\alpha q} \psi_{qr} \right)^2 = 0. \quad (2.12)$$

Here $\{\psi_{jl}\}_{j,l=1}^{n,2}$ are the Grassmann variables (n variables for each determinant in (1.7)) and

$$d\Psi_{s,l} = \prod_{r=1}^s \prod_{j=1}^l d\bar{\psi}_{jr} d\psi_{jr}. \quad (2.13)$$

In view of (1.2) and (2.5) we get

$$\begin{aligned} \mathbf{E} \left\{ \prod_{r=1}^2 \left(1 - \frac{1}{n} \sum_{p,q=1}^n \bar{a}_{\alpha p} a_{\alpha q} \bar{\psi}_{pr} \psi_{qr} \right) \right\} &= 1 - \frac{1}{n} \sum_{r=1}^2 \sum_{p,q=1}^n \mathbf{E} \{ \bar{a}_{\alpha p} a_{\alpha q} \} \bar{\psi}_{pr} \psi_{qr} \\ &\quad + \frac{1}{n^2} \sum_{p_1, q_1=1}^n \sum_{p_2, q_2=1}^n \mathbf{E} \{ \bar{a}_{\alpha p_1} a_{\alpha q_1} \bar{a}_{\alpha p_2} a_{\alpha q_2} \} \bar{\psi}_{p_1 1} \psi_{q_1 1} \bar{\psi}_{p_2 2} \psi_{q_2 2} \\ &= 1 - \frac{1}{n} \sum_{r=1}^2 \sum_{p=1}^n \bar{\psi}_{pr} \psi_{pr} + \frac{1}{n^2} \sum_{p \neq q} \bar{\psi}_{p1} \psi_{p1} \bar{\psi}_{q2} \psi_{q2} - \frac{1}{n^2} \sum_{p \neq q} \bar{\psi}_{p1} \psi_{p2} \bar{\psi}_{q2} \psi_{q1} \\ &\quad + \frac{2\mu_4 + 1/2}{n^2} \sum_{p=1}^n \bar{\psi}_{p1} \psi_{p1} \bar{\psi}_{p2} \psi_{p2} = \det Q_2^{(n)} + \frac{2\kappa_4}{n^2} \sum_{p=1}^n \bar{\psi}_{p1} \psi_{p1} \bar{\psi}_{p2} \psi_{p2}, \end{aligned} \quad (2.14)$$

where $\Psi_s^{(l)}$ and $Q_s^{(l)}$ are the matrix with Grassmann entries

$$\Psi_s^{(l)} = \left\{ \sum_{p=1}^l \bar{\psi}_{pr} \psi_{pt} \right\}_{r,t=1}^s, \quad Q_s^{(l)} = 1 - n^{-1} \Psi_s^{(l)}, \quad (2.15)$$

μ_4 is the 4-th moment of the probability law of $\Im a_{\alpha j}$, $\Re a_{\alpha j}$ of (1.2), and κ_4 is defined in (1.10).

Thus, (2.11) and (2.14) yield

$$\begin{aligned} F_2(\Lambda_2) &= \int e^{\text{tr} \Psi_2^{(n)} \Lambda_2} \left(\det Q_2^{(n)} + \frac{2\kappa_4}{n^2} \sum_{p=1}^n \bar{\psi}_{p1} \psi_{p1} \bar{\psi}_{p2} \psi_{p2} \right)^m d\Psi_{2,n} \\ &= \sum_{q=1}^m \binom{m}{q} \frac{(2\kappa_4)^q}{n^{2q}} \int e^{\text{tr} \Psi_2^{(n)} \Lambda_2} \det^{m-q} Q_2^{(n)} \left(\sum_{p=1}^n \bar{\psi}_{p1} \psi_{p1} \bar{\psi}_{p2} \psi_{p2} \right)^q d\Psi_{2,n} \\ &= \sum_{q=1}^m \binom{m}{q} \frac{n!}{(n-q)!} \frac{(2\kappa_4)^q}{n^{2q}} \int e^{\text{tr} \Psi_2^{(n-q)} \Lambda_2} \det^{m-q} Q_2^{(n-q)} d\Psi_{2,n-q} \\ &=: \sum_{q=1}^m \binom{m}{q} \frac{n!}{(n-q)!} \frac{(2\kappa_4)^q}{n^{2q}} I_{2,q}. \end{aligned} \quad (2.16)$$

Here we used the symmetry of $\bar{\psi}_{lp}$, ψ_{lp} and

$$\int \bar{\psi}_{p1} \psi_{p1} \bar{\psi}_{p2} \psi_{p2} f(\bar{\psi}_{p1}, \psi_{p1}, \bar{\psi}_{p2}, \psi_{p2}) d\bar{\psi}_{p1} d\psi_{p1} d\bar{\psi}_{p2} d\psi_{p2} = f(0, 0, 0, 0).$$

To compute $I_{2,q}$ we use the following lemma

Lemma 1. *Let A be any $p \times p$ matrix and let l be a positive integer. Then we have*

$$\det^l A = K_{p,l} \int \frac{e^{\text{tr } AU}}{\det^{p+l} U} d\mu(U), \quad (2.17)$$

where

$$K_{p,l} = (-1)^{p(p-1)/2} S_p^{-1} \prod_{s=0}^{p-1} (l+s)!, \quad S_p = \prod_{s=1}^p s!, \quad (2.18)$$

U is a unitary matrix with eigenvalues $\{u_j\}_{j=1}^p$, W is a matrix which diagonalizes U and

$$d\mu(U) = \Delta^2(u_1, \dots, u_p) dW \prod_{j=1}^p \frac{du_j}{2\pi i}, \quad (2.19)$$

where du_j means the integration over the circle $\omega = \{z : |z| = 1\}$, dW is the Haar measure over the unitary group $U(p)$, and $\Delta(u_1, \dots, u_p)$ is the Vandermonde determinant of u_j -s.

Remark 1. 1. Lemma 1 is a particular case of the superbosonization formula which was proved in the physics paper [16]. We give below (see Subsection 2.4) a different proof for this simple case.

2. Since both sides of (2.17) are analytic functions of $a_{i,j}$, we can take A with not necessary complex but also with even Grassmann elements.

3. Combining (2.17) and (2.9) we get that for any $p \times p$ matrix A

$$\int e^{\text{tr } A \Psi^{(l)}} d\Psi_{p,l} = K_{p,l} \int \frac{e^{\text{tr } AU}}{\det^{p+l} U} d\mu(U), \quad (2.20)$$

where $\Psi^{(l)} = \{\sum_{s=1}^l \psi_{sj} \psi_{sr}\}_{j,r=1}^p$ and $d\Psi_{p,l}$ is defined in (2.13).

Using Lemma 1 and Remark 1.3, we obtain

$$\begin{aligned} I_{2,q} &= K_{2,m-q} \int \frac{e^{\text{tr } \Lambda_2 \Psi_2^{(n-q)} + \text{tr } W_2 - n^{-1} \text{tr } W_2 \Psi_2^{(n-q)}}}{\det^{m-q+2} W_2} d\Psi_{2,n-q} d\mu(W_2) \\ &= K_{2,m-q} K_{2,n-q} \int \frac{e^{\text{tr } W_2 + \text{tr } \Lambda_2 U_2 - n^{-1} \text{tr } U_2 W_2}}{\det^{m-q+2} W_2 \det^{n-q+2} U_2} d\mu(U_2) d\mu(W_2) \\ &= K_{2,n-q} \int \frac{e^{\text{tr } \Lambda_2 U_2} \det^{m-q}(I - n^{-1} U_2)}{\det^{n-q+2} U_2} d\mu(U_2), \end{aligned} \quad (2.21)$$

where U_2 and W_2 are unitary 2×2 matrices, and $d\mu(U_2)$, $d\mu(W_2)$ are defined in (2.19).

Recall that we are interested in $\Lambda_2 = \Lambda_{0,2} + \hat{\xi}_2/(na)^\beta$, where $\Lambda_{0,2} = \text{diag}\{\lambda_0, \lambda_0\}$, $\hat{\xi}_2 = \text{diag}\{\xi_1, \xi_2\}$, and a, β are defined in (2.1), (2.2). Substituting (2.19) in (2.21) and using that functions $\det(I - n^{-1} U_2)$, $\text{tr } \Lambda_0 U_2$, and $\det U_2$ are unitary invariant, we obtain from (2.21)

$$\begin{aligned} I_{2,q} &= K_{2,n-q} \int \oint_{\omega} e^{\text{tr } \Lambda_{0,2} V_2 + (na)^{-\beta} \text{tr } \hat{\xi}_2 W^* V_2 W} \prod_{r=1}^2 \frac{(1 - \frac{v_r}{n})^{m-q}}{v_r^{n-q+2}} (v_1 - v_2)^2 d\mu(W) \frac{dv_1 dv_2}{(2\pi i)^2} \\ &= \frac{K_{2,n-q}}{n^{2(n-q)}} \int \oint_{\omega} e^{\text{tr } \Lambda_{0,2} V_2 + (na)^{-\beta} \text{tr } \hat{\xi}_2 W^* V_2 W} \prod_{r=1}^2 \frac{(1 - v_r)^{m-q}}{v_r^{n-q+2}} (v_1 - v_2)^2 d\mu(W) \frac{dv_1 dv_2}{(2\pi i)^2}, \end{aligned} \quad (2.22)$$

where ω is any closed contour encircling 0. The integral over the unitary group $U(2)$ can be computed using the Harish Chandra/Itsyson-Zuber formula (see e.g. [14], Appendix 5):

Proposition 2. *Let A be the normal $p \times p$ matrix with distinct eigenvalues $\{a_i\}_{i=1}^p$ and $B = \text{diag}\{b_1, \dots, b_p\}$. Then for any symmetric function $f(B)$ of $\{b_j\}_{j=1}^p$ we have*

$$\int_{U(p)} \int e^{\text{tr} AU^*BU} \Delta^2(B) f(B) dU dB = S_p \int e^{\sum_{j=1}^p a_j b_j} \frac{\Delta(B)}{\Delta(A)} f(b_1, \dots, b_p) dB, \quad (2.23)$$

where S_p is defined in (2.18), $dB = \prod_{j=1}^p db_j$, dU is the Haar measure over the unitary group $U(n)$ and $\Delta(A)$, $\Delta(B)$ are the Vandermonde determinants of the eigenvalues $\{a_i\}_{i=1}^p$, $\{b_i\}_{i=1}^p$ of A and B .

This and formula (2.22) yields

$$I_{2,q} = \frac{2n^{\beta-1} a^\beta K_{2,n-q}}{n^{2(n-q)}} \oint_{\omega} e^{n \text{tr} \Lambda_{0,2} V_2 + n^{1-\beta} a^{-\beta} \text{tr} \hat{\xi}_2 V_2} \prod_{r=1}^2 \frac{(1-v_r)^{m-q}}{v_r^{n-q+2}} \frac{(v_1-v_2) dv_1 dv_2}{(\xi_1 - \xi_2)(2\pi i)^2}. \quad (2.24)$$

Hence, since

$$\frac{n!}{(n-q)!} \cdot \frac{(n-q+1)!(n-q)!}{n^{2n-q}} = 2\pi n^2 e^{-2n} (1 + O(1/n))$$

we get (2.3) for $k=1$ from (2.16), (2.18), and (2.24).

2.3 Asymptotic integral representation for F_{2k}

Using (2.9) and (2.12), we obtain from (1.7) (cf. (2.11))

$$\begin{aligned} F_{2k}(\Lambda_{2k}) &= \mathbf{E} \left\{ \int e^{\sum_{l=1}^{2k} \sum_{p,q=1}^n (\lambda_l - H)_{p,q} \bar{\psi}_{pr} \psi_{qr}} d\Psi_{2k,n} \right\} \\ &= \mathbf{E} \left\{ \int e^{\sum_{s=1}^{2k} \lambda_s \sum_{p=1}^n \bar{\psi}_{ps} \psi_{ps}} \prod_{\alpha=1}^m e^{-\frac{1}{n} \sum_{r=1}^{2k} \left(\sum_{p=1}^n \bar{a}_{\alpha p} \bar{\psi}_{pr} \right) \left(\sum_{q=1}^n a_{\alpha q} \psi_{qr} \right)} d\Psi_{2k,n} \right\} \\ &= \mathbf{E} \left\{ \int e^{\sum_{s=1}^{2k} \lambda_s \sum_{p=1}^n \bar{\psi}_{ps} \psi_{ps}} \prod_{\alpha=1}^m \prod_{r=1}^{2k} \left(1 - \frac{1}{n} \sum_{p,q=1}^n \bar{a}_{\alpha p} a_{\alpha q} \bar{\psi}_{pr} \psi_{qr} \right) d\Psi_{2k,n} \right\}. \end{aligned} \quad (2.25)$$

In view of (1.2) similarly to (2.14) we get

$$\begin{aligned} \mathbf{E} \left\{ \prod_{r=1}^{2k} \left(1 - \frac{1}{n} \sum_{p,q=1}^n \bar{a}_{\alpha p} a_{\alpha q} \bar{\psi}_{pr} \psi_{qr} \right) \right\} &= \det Q_{2k}^{(n)} \\ &+ \frac{2\kappa_4}{n^2} \sum_{l_1 < l_2, s_1 < s_2} \det(Q_{2k}^{(n)})^{(l_1, l_2; s_1, s_2)} \sum_{p=1}^n \bar{\psi}_{pl_1} \psi_{ps_1} \bar{\psi}_{pl_2} \psi_{ps_2} + n^{-2} \Phi(\Psi), \end{aligned} \quad (2.26)$$

where $Q_{2k}^{(n)}$ is defined in (2.15), $\det(Q_{2k}^{(n)})^{(l_1, l_2; s_1, s_2)}$ is $(2k-2) \times (2k-2)$ minor of matrix $Q_{2k}^{(n)}$ without lines s_1, s_2 and columns l_1, l_2 , κ_4 is defined in (1.10) and $\Phi(\Psi)$ is a polynomial of the variables $\{(n^{-1}\Psi_{2k}^{(n)})_{r,s}\}_{r,s=1}^{2k}$ and

$$n^{-1}\sigma_{\bar{l}, \bar{s}}^{(n)} = \frac{1}{n} \sum_{p=1}^n \prod_{j=1}^q \bar{\psi}_{pl_j} \psi_{ps_j}, \quad \bar{l} = (l_1, \dots, l_q), \quad \bar{s} = (s_1, \dots, s_q). \quad (2.27)$$

Now we use

Lemma 2. *Set $A = \{a_{i,j}\}_{i,j=1}^{2k}$, $b = \{b_{\bar{l}, \bar{s}}\}$, where \bar{l}, \bar{s} is defined in (2.27). Let $\Phi_r(A, b)$ be an analytic function of the variables $\{a_{i,j}\}$ and $\{b_{\bar{l}, \bar{s}}\}$ and let $(1-\varepsilon)n < r \leq n$, $0 \leq l < \varepsilon n$ with some sufficiently small $\varepsilon > 0$. Then there exist absolute constants C_0, C_1 such that*

$$\int \Phi_r(n^{-1}\Psi_{2k}^{(r)}, n^{-1}\sigma^{(r)}) \tilde{\mu}_{2k,l}^{(r)}(\Psi) d\Psi_{2k,r} \leq C_0 \max_{|a_{i,j}|, |b_{\bar{l}, \bar{s}}| \leq C_1} |\Phi_r(A, b)| \cdot \int \tilde{\mu}_{2k,l}^{(r)}(\Psi) d\Psi_{2k,r},$$

where

$$\tilde{\mu}_{2k,l}^{(r)}(\Psi) = e^{\text{tr} \Psi_{2k}^{(r)} \Lambda_{2k}} \det^{m-l} Q_{2k}^{(r)}. \quad (2.28)$$

The proof of Lemma 2 is given in Subsection 2.4.

Denote the expression multiplied by κ_4 in the r.h.s. of (2.26) by $n^{-1}X$. Write

$$\begin{aligned} & \left(\det Q_{2k}^{(n)} + \frac{\kappa_4}{n} X + n^{-2} \Phi(\Psi) \right)^m \\ &= \sum_{k_1+k_2 \leq m} \frac{m!}{k_1! k_2! (m-k_1-k_2)!} \left(\det Q_{2k}^{(n)} \right)^{m-k_1-k_2} \left(\frac{\kappa_4}{n} X \right)^{k_1} \left(n^{-2} \Phi(\Psi) \right)^{k_2}. \end{aligned} \quad (2.29)$$

It is easy to see that the terms in (2.29) such that $k_1 + k_2 \geq \varepsilon m$ give the contribution of order $e^{-\varepsilon n \log n}$ and thus can be omitted. Hence, (2.25), (2.26), and Lemma 2 yield

$$\begin{aligned} F_{2k}(\Lambda_{2k}) &= (1 + O(n^{-1})) \int d\Psi_{2k,n} e^{\text{tr} \Psi_{2k}^{(n)} \Lambda_{2k}} \\ & \left(\det Q_{2k}^{(n)} + \frac{2\kappa_4}{n^2} \sum_{l_1 < l_2, s_1 < s_2} \det(Q_{2k}^{(n)})^{(l_1, l_2; s_1, s_2)} \sum_{p=1}^n \bar{\psi}_{pl_1} \psi_{ps_1} \bar{\psi}_{pl_2} \psi_{ps_2} \right)^m, \end{aligned} \quad (2.30)$$

where $Q_{2k}^{(n)}$ and $\Psi_{2k}^{(n)}$ are defined in (2.15).

To compute the r.h.s. of (2.30) we use the Newton binomial formula and observe that the term with $p_s = p_l$ in the product $\prod_{j=1}^q \left(n^{-1} \sum_{p_j} \bar{\psi}_{p_j l_{1,j}} \psi_{p_j s_{1,j}} \bar{\psi}_{p_j l_{2,j}} \psi_{p_j s_{2,j}} \right)$ can be expressed in terms of (2.27) with an additional factor n^{-1} . Therefore, according to Lemma 2 it suffices to consider only the terms with $p_s \neq p_l$ or, taking into account the symmetry, the term $p_1 = n, p_2 = n-1, \dots, p_q = n-q+1$ with coefficient $n!/(n-q)!$. Thus, we can write

$$F_{2k}(\Lambda_{2k}) = (1 + O(n^{-1})) \sum_{q=0}^m \binom{m}{q} \frac{n!}{(n-q)!} \frac{(2\kappa_4)^q}{n^q} I_{2k,q}, \quad (2.31)$$

where

$$I_{2k,q} = \int \tilde{\mu}_{2k,q}^{(n)}(\Psi) P_{q,n}^{(n)}(\Psi) d\Psi_{2k,n}, \quad (2.32)$$

$$P_{q,l}^{(r)}(\Psi) = \prod_{p=n-q+1}^l \left(\sum_{l_1^p < l_2^p, s_1^p < s_2^p} \det(Q_{2k}^{(r)})^{(l_1^p, l_2^p; s_1^p, s_2^p)} \bar{\psi}_{pl_1^p} \psi_{ps_1^p} \bar{\psi}_{pl_2^p} \psi_{ps_2^p} \right), \quad (2.33)$$

where $\tilde{\mu}_{2k,q}^{(n)}$ is defined in (2.28). Note that

$$\det(Q_{2k}^{(n)})^{(l_1, l_2; s_1, s_2)} = \det(Q_{2k}^{(n-q)})^{(l_1, l_2; s_1, s_2)} + n^{-1} \tilde{\Phi}_1(\Psi),$$

$\tilde{\Phi}_1(\Psi)$ are polynomials of $\{\psi_{js}\}_{j,s=1}^{n-2k}$, $\{\bar{\psi}_{js}\}_{j,s=1}^{n-2k}$ with the sum of the coefficients of order $O(q)$, $m, n \rightarrow \infty$. Hence, using Lemma 2, we get

$$I_{2k,q} = \int \tilde{\mu}_{2k,q}^{(n)}(\Psi) P_{q,n}^{(n-q)}(\Psi) d\Psi_{2k,n} (1 + O(q/n)) =: \tilde{I}_{2k,q} (1 + O(q/n)), \quad (2.34)$$

According to Lemma 1 and (2.10) we can rewrite $\tilde{I}_{2k,q}$ as

$$\begin{aligned} \tilde{I}_{2k,q} &= K_{2k,m-q} \int d\mu(V) d\Psi_{2k,n-q} \frac{e^{\text{tr } \Lambda_{2k} \Psi_{2k}^{(n-q)} + \text{tr } Q_{2k}^{(n-q)} V}}{\det^{m-q+2k} V} \\ &\times \int \prod_{p=n-q+1}^n \prod_{l=1}^{2k} d\bar{\psi}_{pl} d\psi_{pl} e^{\sum_{i,j=1}^{2k} (\Lambda_{2k} - n^{-1}V)_{i,j} \sum_{p=n-q+1}^n \bar{\psi}_{pi} \psi_{pj}} P_{q,n}^{(n-q)}(\Psi) \\ &= K_{2k,m-q} \int d\mu(V) d\Psi_{2k,n-q} \frac{e^{\text{tr } \Lambda_{2k} \Psi_{2k}^{(n-q)} + \text{tr } Q_{2k}^{(n-q)} V}}{\det^{m-q+2k} V} \\ &\times \left(\sum_{l_1 < l_2, s_1 < s_2} \det(Q_{2k}^{(n-q)})^{(l_1, l_2; s_1, s_2)} \det(\Lambda_{2k} - n^{-1}V)_{(l_1, l_2; s_1, s_2)} \right)^q. \end{aligned} \quad (2.35)$$

Besides, the Cauchy-Binet formula (see [9]) yields for $2k \times 2k$ matrices A, B

$$\sum_{l_1 < l_2, s_1 < s_2} \det A^{(l_1, l_2; s_1, s_2)} B_{(l_1, l_2; s_1, s_2)} = \frac{1}{2} \frac{d^2}{dx^2} \det(x - AB) \Big|_{x=0}.$$

Thus, using again Lemma 1 and Remark 1.3, we obtain

$$\begin{aligned}
I_{2k,q} &= K_{2k,m-q} \int d\mu(V) d\Psi_{2k,n-q} \frac{e^{\text{tr } \Lambda_{2k} \Psi_{2k}^{(n-q)} + \text{tr } Q_{2k}^{(n-q)} V}}{\det^{m-q+2k} V} \\
&\quad \int \prod_{s=1}^q \frac{dz_s}{2\pi i z_s^3} \int \prod_{s=1}^q \frac{K_{2k,1} d\mu(W_s)}{\det^{2k+1} W_s} e^{\text{tr } Q_{2k}^{(n-q)} (\Lambda_{2k} - n^{-1} V) \sum_{p=1}^q W_p - \sum_{p=1}^q z_p \text{tr } W_p} \quad (2.36) \\
&= K_{2k,m-q} K_{2k,n-q} \int d\mu(V) d\mu(U) \frac{e^{\text{tr } \Lambda_{2k} U + \text{tr } (1-n^{-1} U) V}}{\det^{m-q+2k} V \det^{n-q+2k} U} \\
&\quad \int \prod_{s=1}^q \frac{dz_s}{2\pi i z_s^3} \int \prod_{s=1}^q \frac{K_{2k,1} d\mu(W_s)}{\det^{2k+1} W_s} e^{\text{tr } (1-n^{-1} U) (\Lambda_{2k} - n^{-1} V) \sum_{p=1}^q W_p - \sum_{p=1}^q z_p \text{tr } W_p} \\
&= K_{2k,n-q} \int \frac{e^{\text{tr } \Lambda_{2k} U} \det^{m-q} (1 - n^{-1} U) \det^{m-q} (1 - n^{-1} \sum_{p=1}^q W_p)}{\det^{n-q+2k} U} \\
&\quad \times e^{\text{tr } (1-n^{-1} U) \Lambda_{2k} \sum_{p=1}^q W_p - \sum_{p=1}^q z_p \text{tr } W_p} \prod_{s=1}^q \frac{dz_s}{2\pi i z_s^3} \prod_{s=1}^q \frac{K_{2k,1} d\mu(W_s)}{\det^{2k+1} W_s} d\mu(U).
\end{aligned}$$

Besides,

$$\det^{m-q} (1 - n^{-1} \sum_{p=1}^q W_p) = e^{-c_{m,n} \sum_{p=1}^q \text{tr } W_p} (1 + O(q/n)).$$

Substituting this to (2.36) and using (2.17), we get

$$\begin{aligned}
\tilde{I}_{2k,q} &= (1 + O(q/n)) K_{2k,n-q} \int \frac{e^{\text{tr } \Lambda_{2k} U} \det^{m-q} (1 - n^{-1} U)}{\det^{n-q+2k} U} \\
&\quad \times e^{\text{tr } (1-n^{-1} U) \Lambda_{2k} \sum_{p=1}^q W_p - \sum_{p=1}^q (z_p + c_{m,n}) \text{tr } W_p} \prod_{s=1}^q \frac{dz_s}{2\pi i z_s^3} \prod_{s=1}^q \frac{K_{2k,1} d\mu(W_s)}{\det^{2k+1} W_s} d\mu(U) \quad (2.37) \\
&= (1 + O(q/n)) K_{2k,n-q} \int \frac{e^{\text{tr } \Lambda_{2k} U} \det^{m-q} (1 - n^{-1} U)}{\det^{n-q+2k} U} S_2^q((1 - n^{-1} U) \Lambda_{2k}) d\mu(U)
\end{aligned}$$

Recall that we are interested in $\Lambda_{2k} = \Lambda_{0,2k} + \hat{\xi}/(na)^\beta$, where $\Lambda_0 = \text{diag}\{\lambda_0, \dots, \lambda_0\}$, $\hat{\xi} = \text{diag}\{\xi_1, \dots, \xi_{2k}\}$, and a, β are defined in (2.1), (2.2). Thus,

$$\begin{aligned}
\tilde{I}_{2k,q} &= (1 + O(n^{-\beta}) + O(q/n)) K_{2k,n-q} \int \frac{e^{\text{tr } \Lambda_{2k} U} \det^{m-q} (1 - n^{-1} U)}{\det^{n-q+2k} U} \\
&\quad \times S_2^q((1 - n^{-1} U) \Lambda_0) d\mu(U), \quad (2.38)
\end{aligned}$$

Let us change variables to $U = W^* V W$, where W is a unitary $2k \times 2k$ matrix and $V = \text{diag}\{v_1, \dots, v_{2k}\}$. Since functions $\det(I - n^{-1} U)$, $S_2((I - n^{-1} U) \Lambda_0)$, and $\det U$ are unitary invariant, (2.38) implies

$$\begin{aligned}
I_{2k,q} &= (1 + O(n^{-\beta}) + O(q/n)) K_{2k,n-q} \oint_{\omega} \prod_{j=1}^{2k} \frac{dv_j}{2\pi i} \int d\mu(W) e^{\text{tr } W^* V W \Lambda_{2k}} \Delta^2(V) \\
&\quad \frac{\det^{m-q} (I - n^{-1} V)}{\det^{n-q+2k} V} S_2^q((I - n^{-1} V) \Lambda_0). \quad (2.39)
\end{aligned}$$

where ω is any closed contour encircling 0. The integral over the unitary group $U(2k)$ can be computed using the Harish Chandra/Itsyson-Zuber formula (2.23). Shifting $v_i \rightarrow nv_i$, we obtain

$$\begin{aligned} \tilde{I}_{2k,q} &= \frac{S_{2k} K_{2k,n-q} (n^{\beta-1} a^\beta)^{k(2k-1)}}{n^{2k(n-q)} D_{2k}} \oint_{\omega} \prod_{j=1}^{2k} \frac{dv_j}{2\pi i} e^{n \operatorname{tr} V \Lambda_0 + n^{1-\beta} a^{-\beta} \operatorname{tr} V \hat{\xi}} \frac{\Delta(V)}{\Delta(\hat{\xi})} \\ &\quad \prod_{l=1}^{2k} \frac{(1-v_l)^{m-q}}{v_l^{n-q+2k}} S_2^q(I-V)(1+O(n^{-\beta})+O(q/n)). \end{aligned} \quad (2.40)$$

Hence, since

$$\frac{n!}{(n-q)!} \cdot \frac{\prod_{s=0}^{2k-1} (n-q+s)!}{n^{2k(n-q)+q}} = (2\pi)^k n^{2k^2} e^{-2kn} (1+O(1/n)),$$

we get (2.3) from (2.31), (2.34), and (2.40).

2.4 Proofs of Lemmas 1, 2

Proof of Lemma 1 Let A be a normal matrix. Then we can set $A = V_0^* A_0 V_0$ and $U = W^* U_0 W$, where $A_0 = \operatorname{diag}(a_1, \dots, a_p)$, $U_0 = \operatorname{diag}(u_1, \dots, u_p)$ and V_0, W are the matrices diagonalizing A and U correspondingly. We obtain

$$I := \int \frac{e^{\operatorname{tr} AU}}{\det^{p+l} U} dU = \int \frac{e^{\operatorname{tr} V_0^* A_0 V_0 W^* U_0 W} \Delta^2(u_1, \dots, u_p)}{\prod_{j=1}^p u_j^{p+l}} d\mu(W) \prod_{j=1}^p \frac{du_j}{2\pi i}.$$

Shifting integration with respect to W as $WV_0^* \rightarrow W$ and using (2.23), we obtain

$$\begin{aligned} I &= q_p \oint_{\omega} \frac{e^{\sum_{j=1}^p a_j u_j} \Delta(u_1, \dots, u_p)}{\Delta(A_0) \prod_{j=1}^p u_j^{p+l}} \prod_{j=1}^p \frac{du_j}{2\pi i} \\ &= \frac{q_p}{\Delta(A_0)} \det \left[\oint_{\omega} \frac{e^{a_j u_j}}{u_j^{p+l-s}} \frac{du_j}{2\pi i} \right]_{j,s=1,0}^{p,p-1} = \frac{q_p}{\Delta(A_0)} \det \left[\frac{a_j^{p+l-s-1}}{(p+l-s-1)!} \right]_{j,s=1,0}^{p,p-1} \\ &= \frac{q_p \Delta(1/a_1, \dots, 1/a_p) \prod_{j=1}^p a_j^{p+l-1}}{\prod_{s=0}^{p-1} (p+l-s-1)! \Delta(A_0)} = \frac{(-1)^{\frac{p(p-1)}{2}} q_p \prod_{j=1}^p a_j^l}{\prod_{s=0}^{p-1} (p+l-s-1)!} = \frac{(-1)^{\frac{p(p-1)}{2}} q_p \det^l A}{\prod_{s=0}^{p-1} (p+l-s-1)!}, \end{aligned}$$

and (2.17) is proved for the normal A .

Let now A be an arbitrary matrix. According to the polar decomposition, we can write $A = SW$, where W is a unitary $p \times p$ matrix and S is a diagonal $p \times p$ matrix. Since we proved (2.17) for any normal A , we proved it for $S = \operatorname{diag}\{e^{i\alpha_1}, \dots, e^{i\alpha_p}\}$, $\alpha_1, \dots, \alpha_p \in \mathbb{R}$. Besides, it is easy to see that both sides of (2.17) is analytic functions of the elements of S . Therefore, we are proved (2.17) for any A .

□

Proof of Lemma 2 According to Lemma 1 and (2.28), we have

$$\begin{aligned} \int \tilde{\mu}_{2k,l}^{(r)}(\Psi) d\Psi_{2k,r} &= K_{2k,m-l} \int \int \frac{e^{\text{tr} \Lambda_{2k} \Psi_{2k}^{(r)} + \text{tr}(1-n^{-1}\Psi_{2k}^{(r)})V}}{\det^{m-l+2k} V} d\mu(V) d\Psi_{2k,r} \\ &= K_{2k,m-l} \int \frac{e^{\text{tr} V \det^r(\Lambda_{2k} - n^{-1}V)}}{\det^{m-l+2k} V} d\mu(V) =: J. \end{aligned} \quad (2.41)$$

It is proved below (see Section 3 and 4 (note that if we change $v \rightarrow \lambda_0(1-v)$ in J we obtain the integral like in (3.3) and (4.1))) that

$$|J| \geq \frac{CK_{2k,m-l}}{n^{2k(m-l)}} \oint_{\tilde{\omega}} e^{n \sum_{j=1}^{2k} \Re v_j + n^{1-\beta} a^{-\beta} \sum_{j=1}^{2k} \xi_j \Re v_j} \prod_{j=1}^{2k} |\lambda_0 - v_j|^r |\Delta(V)|^2 \prod_{j=1}^{2k} \frac{|dv_j|}{|v_j|^{m-l+2k}}, \quad (2.42)$$

where a and β are defined in (2.1) and (2.2) and

$$\tilde{\omega} = \left\{ z \in \mathbb{C} : |z| = \left(\frac{m-l}{n} \lambda_0 \right)^{1/2} \right\}. \quad (2.43)$$

Moreover the integral outside of the any n -independent neighborhood of $v_{\pm} = (\lambda_0 + \frac{m-l}{n} - 1)/2 \pm \pi \lambda_0 \rho(\lambda_0)$ give contribution $O(e^{-nC})$, hence we can deform $\tilde{\omega}$ near $z = (\frac{m-l}{n} \lambda_0)^{1/2}$ such that $|v - \lambda_0| > \delta$ on $\tilde{\omega}$. Thus, if we define

$$\langle (\dots) \rangle = J^{-1} \int (\dots) \tilde{\mu}_{2k,l}^{(r)}(\Psi) d\Psi_{2k,r} \quad (2.44)$$

the definition is correct.

Using (2.6), we get

$$\begin{aligned} \langle \Phi_r \rangle &:= \langle \Phi_r(n^{-1}\Psi_{2k}^{(r)}, \sigma^{(r)}) \rangle = \oint_{\Omega} \Phi_r(A, b) \prod_{i,j=1}^{2k} \frac{da_{i,j}}{2\pi i} \prod_{\bar{l}, \bar{s}} \frac{db_{\bar{l}, \bar{s}}}{2\pi i} \\ &\times \left\langle \prod_{i,j=1}^{2k} \frac{1}{a_{i,j} - n^{-1}(\Psi_{2k}^{(r)})_{i,j}} \prod_{\bar{l}, \bar{s}} \frac{1}{b_{\bar{l}, \bar{s}} - n^{-1}\sigma_{\bar{l}, \bar{s}}^{(r)}} \right\rangle. \end{aligned} \quad (2.45)$$

Thus, to prove Lemma 2, we have to estimate the expectation above. Expanding the functions into the series with respect to $\{\Psi_{2k}^{(r)}\}_{i,j}$, $\{\sigma_{\bar{l}, \bar{s}}^{(r)}\}$, we get

$$\begin{aligned} &\left\langle \prod_{i,j=1}^{2k} \frac{1}{a_{i,j} - n^{-1}(\Psi_{2k}^{(r)})_{i,j}} \prod_{\bar{l}, \bar{s}} \frac{1}{b_{\bar{l}, \bar{s}} - n^{-1}\sigma_{\bar{l}, \bar{s}}^{(r)}} \right\rangle \\ &= \sum_{i,j, \bar{l}, \bar{s}}^r \sum_{l_{i,j}=1}^r \sum_{t_{\bar{l}, \bar{s}}=1}^r \left\langle \prod_{i,j=1}^{2k} (n^{-1}(\Psi_{2k}^{(r)})_{i,j})^{l_{i,j}} \prod_{\bar{l}, \bar{s}} (n^{-1}\sigma_{\bar{l}, \bar{s}}^{(r)})^{t_{\bar{l}, \bar{s}}} \right\rangle \prod_{i,j=1}^{2k} a_{i,j}^{-l_{i,j}-1} \prod_{\bar{l}, \bar{s}} b_{\bar{l}, \bar{s}}^{-t_{\bar{l}, \bar{s}}-1} \\ &:= \sum_{i,j, \bar{l}, \bar{s}}^r \sum_{l_{i,j}=1}^r \sum_{t_{\bar{l}, \bar{s}}=1}^r M(\{l_{i,j}\}, \{t_{\bar{l}, \bar{s}}\}) \prod_{i,j=1}^{2k} a_{i,j}^{-l_{i,j}-1} \prod_{\bar{l}, \bar{s}} b_{\bar{l}, \bar{s}}^{-t_{\bar{l}, \bar{s}}-1}. \end{aligned} \quad (2.46)$$

To estimate the moments $\{M(\{l_{i,j}\}, \{t_{\bar{l},\bar{s}}\})\}$, we introduce the generating function

$$F(\zeta, z) := \left\langle \exp \left\{ n^{-1} \text{tr} \zeta \Psi_{2k}^{(r)} + n^{-1} \sum_{\bar{l}, \bar{s}} z_{\bar{l}, \bar{s}} \sigma_{\bar{l}, \bar{s}}^{(r)} \right\} \right\rangle, \quad (2.47)$$

where $\zeta = \{\zeta_{i,j}\}_{i,j=1}^{2k}$. It is easy to see that the derivatives $F(\zeta, z)$ with respect to $\{\zeta_{i,j}\}$ and $\{z_{\bar{l},\bar{s}}\}$ will give us the moments $\{M(\{l_{i,j}\}, \{t_{\bar{l},\bar{s}}\})\}$.

Using Lemma 1 and then integrating over $d\Psi_{2k,r}$, we obtain

$$\begin{aligned} F(\zeta, z) &= \frac{K_{2k,m-l}}{J} \int \frac{e^{\text{tr} V + \text{tr} (\Lambda_{2k} + n^{-1} \zeta - n^{-1} V) \Psi_{2k}^{(r)} + n^{-1} \sum_{\bar{l}, \bar{s}} z_{\bar{l}, \bar{s}} \sigma_{\bar{l}, \bar{s}}^{(r)}}}{\det^{m-l+2k} V} d\mu(V) d\Psi_{2k,r} \\ &= \frac{K_{2k,m-l}}{J} \int \frac{e^{\text{tr} V}}{\det^{m-l+2k} V} \Phi_1^r(\Lambda_{2k} - n^{-1} V, n^{-1} \zeta, n^{-1} z) d\mu(V), \end{aligned}$$

where J is defined in (2.41). Moreover, according to (2.9) – (2.10), $\Phi_1(\Lambda_{2k} - n^{-1} V, n^{-1} \zeta, n^{-1} z)$ is a polynomial of the entries of $\Lambda_{2k} - n^{-1} V$ and of $\{\zeta_{i,j}\}$, $\{z_{\bar{l},\bar{s}}\}$ with n -independent coefficients and degree at most $2k$ and such that the degree of each variable in $\Phi_1(V, n^{-1} \zeta, n^{-1} z)$ is at most one. Here we also used that the integral over $d\Psi_{2k,r}$ can be factorized in $\{\bar{\psi}_{pi} \psi_{pj}\}_{i,j=1}^{2k}$. Besides,

$$\Phi_1(n^{-1} V, n^{-1} \zeta, n^{-1} z) = \det(\Lambda_{2k} - n^{-1} V) + \tilde{f}(\Lambda_{2k} - n^{-1} V, n^{-1} \zeta, n^{-1} z), \quad (2.48)$$

where $\tilde{f}(\Lambda_{2k} - n^{-1} V, n^{-1} \zeta, n^{-1} z)$ contains all terms of Φ_1 which includes $\{\zeta_{i,j}\}$ or $\{z_{\bar{l},\bar{s}}\}$.

Recall that we are interested in $\Lambda = \Lambda_0 + \hat{\xi}/(na)^\beta$, where $\Lambda_0 = \text{diag}\{\lambda_0, \dots, \lambda_0\}$, $\hat{\xi} = \text{diag}\{\xi_1, \dots, \xi_{2m}\}$, and a and β are defined in (2.1) and (2.2). Change the variables $v_j \rightarrow nv_j$, $j = 1, \dots, 2k$, where $\{v_j\}$ are the eigenvalues of V , and replace the integration over the unit circle by the integration over $\tilde{\omega}$ of (2.43)

$$F(\zeta, z) = \frac{K_{2k,m-l}}{J \cdot n^{2k(m-l)}} \int \frac{e^{n \text{tr} V}}{\det^{m-l+2k} V} \Phi_1^r(\Lambda_{2k} - V, n^{-1} \zeta, n^{-1} z) d\mu(V) \quad (2.49)$$

We have from the description of Φ_1 and (2.48)

$$|\Phi_1(\Lambda_{2k} - V, n^{-1} \zeta, n^{-1} z)| \leq C |\det(\Lambda_0 - V)| \prod_{i,j=1}^{2k} \left(1 + \frac{C(V) |\zeta_{i,j}|}{n}\right) \prod_{\bar{l}, \bar{s}} \left(1 + \frac{C(V) |z_{\bar{l},\bar{s}}|}{n}\right), \quad (2.50)$$

where $C(V) > 0$ is bounded for $v_j \in \tilde{\omega}$ with $\tilde{\omega}$ of (2.43) (recall that we can deform $\tilde{\omega}$ near $z = \left(\frac{m-l}{n} \lambda_0\right)^{1/2}$ such that $|v - \lambda_0| > \delta$ on $\tilde{\omega}$). Since $\{M(\{l_{i,j}\}, \{t_{\bar{l},\bar{s}}\})\}$ are the derivatives of the generating function, we can write

$$M(\{l_{i,j}\}, \{t_{\bar{l},\bar{s}}\}) = \left\langle \prod_{i,j=1}^{2k} \oint_{\Omega_{i,j}} \frac{l_{i,j}!}{2\pi i} \frac{d\zeta_{i,j}}{\zeta_{i,j}^{l_{i,j}+1}} \prod_{\bar{l}, \bar{s} \in \Sigma_{\bar{l}, \bar{s}}} \oint \frac{t_{\bar{l},\bar{s}}!}{2\pi i} \frac{dz_{\bar{l},\bar{s}}}{z_{\bar{l},\bar{s}}^{t_{\bar{l},\bar{s}}+1}} F(\zeta, z) \right\rangle. \quad (2.51)$$

This, (2.49), and (2.50) yield

$$|M(\{l_{i,j}\}, \{t_{\bar{l},\bar{s}}\})| \leq \prod_{i,j=1}^{2k} \min_{t \in \Omega_{i,j}} l_{i,j}! e^{C|t| - l_{i,j} \log |t|} \prod_{\bar{l}, \bar{s}} \min_{t \in \Sigma_{\bar{l}, \bar{s}}} t_{\bar{l},\bar{s}}! e^{C|t| - t_{\bar{l},\bar{s}} \log |t|} \quad (2.52)$$

Choose $\Omega_{i,j} = \{\zeta \in \mathbb{C} : |\zeta| = l_{i,j}/C\}$, $\Sigma_{\vec{l},\vec{s}} = \{z \in \mathbb{C} : |z| = t_{\vec{l},\vec{s}}/C\}$. Then (2.52) yields

$$\begin{aligned} |M(\{l_{i,j}\}, \{t_{\vec{l},\vec{s}}\})| &\leq \prod_{i,j=1}^{2k} \sqrt{2\pi l_{i,j}} C^{l_{i,j}} \prod_{\vec{l},\vec{s}} \sqrt{2\pi t_{\vec{l},\vec{s}}} C^{t_{\vec{l},\vec{s}}} \\ &= \prod_{i,j=1}^{2k} \sqrt{2\pi l_{i,j}} C^{l_{i,j}} \prod_{\vec{l},\vec{s}} \sqrt{2\pi t_{\vec{l},\vec{s}}} C^{t_{\vec{l},\vec{s}}}. \end{aligned} \quad (2.53)$$

Thus, if $|a_{i,j}| > C$ and $|b_{\vec{l},\vec{s}}| > C$ in (2.46) we obtain Lemma 2 from (2.41), (2.45) – (2.46). \square

3 Asymptotic analysis in the bulk of the spectrum.

In this section we prove Theorem 1, passing to the limit $m, n \rightarrow \infty$ in (2.40) for $\lambda_j = \lambda_0 + \xi_j/n\rho(\lambda_0)$, where ρ is defined in (1.5), $\lambda_0 \in \sigma$ with σ of (1.6), and $\xi_j \in [-M, M] \subset \mathbb{R}$, $j = 1, \dots, 2k$.

To this end consider the function

$$V(v, \lambda_0) = -\lambda_0 v - c_{m,n} \log(1 - v) + \log v + S^*, \quad (3.1)$$

where

$$c_{m,n} = \frac{m}{n}, \quad S^* = \frac{\lambda_0 - c_{m,n} + 1}{2} + \frac{c_{m,n}}{2} \log \frac{c_{m,n}}{\lambda_0} - \frac{1}{2} \log \frac{1}{\lambda_0}. \quad (3.2)$$

Then (2.3) and (2.40) yield

$$D_{2k}^{-1}(\lambda_0) n^{-k^2} F_{2k}(\Lambda_{2k}) = Z_{2k} \oint_{\omega_0} W_n(v_1, \dots, v_{2k}) \prod_{j=1}^{2k} dv_j (1 + o(1)), \quad (3.3)$$

where D_{2k} is defined in (1.14),

$$\begin{aligned} W_n(v_1, \dots, v_{2k}) &= e^{-n \sum_{l=1}^{2k} V(v_l, \lambda_0) + \sum_{l=1}^{2k} \frac{\xi_l}{\rho(\lambda_0)} v_l} \frac{\Delta(V)}{\Delta(\widehat{\xi})} \prod_{j=1}^{2k} \frac{1}{v_j^{2k}} \\ &\times \exp \left\{ 2c_{m,n} \kappa_4 S_2((I - V)\Lambda_0) \prod_{l=1}^{2k} \frac{v_l}{1 - v_l} \right\}, \end{aligned} \quad (3.4)$$

and

$$Z_{2k} = \frac{n^{k^2} \rho(\lambda_0)^{k(k-1)} e^{-2k\kappa_4 - \alpha(\lambda_0) \sum_{j=1}^{2k} \xi_j}}{2^{2k} \pi^{2k} c^{k/2}}. \quad (3.5)$$

Now we need the following lemma

Lemma 3. *The function $\Re V(v, \lambda_0)$ for $v = \lambda_0^{-1/2} e^{i\varphi}$, $\varphi \in (-\pi, \pi]$ attains its minimum at*

$$v = v_{\pm} := \lambda_0^{-1/2} e^{\pm i\varphi_0} := \frac{\lambda_0 - c_{m,n} + 1}{2\lambda_0} \pm i\pi\rho(\lambda_0). \quad (3.6)$$

Moreover, if $\varphi \notin U_n(\pm\varphi_0) := (\pm\varphi_0 - n^{-1/2}\log n, \pm\varphi_0 + n^{-1/2}\log n)$, then we have for sufficiently big n

$$\Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0) \geq \frac{C \log^2 n}{n}. \quad (3.7)$$

Proof. Note that for $\varphi \in (-\pi, \pi]$

$$\Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0) = -\lambda_0^{1/2} \cos \varphi - \frac{c_{m,n}}{2} \log \left(1 + \lambda_0^{-1} - 2\lambda_0^{-1/2} \cos \varphi \right) + \log \lambda_0^{-1} + S^*, \quad (3.8)$$

where S^* and $c_{m,n}$ are defined in (3.2). Thus

$$\begin{aligned} \frac{d}{d\varphi} \Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0) &= \lambda_0^{1/2} \sin \varphi \left(1 - \frac{c_{m,n}/\lambda_0}{1 + \lambda_0^{-1} - 2\lambda_0^{-1/2} \cos \varphi} \right), \\ \frac{d^2}{d\varphi^2} \Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0) &= \lambda_0^{1/2} \cos \varphi \left(1 - \frac{c_{m,n}/\lambda_0}{1 + \lambda_0^{-1} - 2\lambda_0^{-1/2} \cos \varphi} \right) \\ &\quad + \frac{2c_{m,n} \sin^2 \varphi / \lambda_0}{(1 + \lambda_0^{-1} - 2\lambda_0^{-1/2} \cos \varphi)^2}, \end{aligned} \quad (3.9)$$

and $\varphi = \pm\varphi_0$ of (3.6) are the minimum points of $\Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0)$. Writing

$$V_{\pm} := V(v_{\pm}, \lambda_0) = \mp i \lambda_0^{-1/2} \sin \varphi_0 \pm i \varphi_0 \pm i c_{m,n} \arcsin \frac{\lambda_0^{-1/2} \sin \varphi_0}{1 + \lambda_0^{-1} - 2\lambda_0^{-1/2} \cos \varphi_0}, \quad (3.10)$$

we conclude that

$$\Re V(v_{\pm}, \lambda_0) = 0.$$

Expanding $\Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0)$ into the Taylor series and using (3.9) – (3.10), we obtain for $\varphi \in U_n(\pm\varphi_0)$:

$$\Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda_0) = \frac{(\pi \lambda_0 \rho(\lambda_0))^2}{c_{m,n}} (\varphi \mp \varphi_0)^2 + O(n^{-3/2} \log^3 n), \quad (3.11)$$

where φ_0 is defined in (3.6). This and $c_{m,n} \rightarrow c$, $m, n \rightarrow \infty$ imply for $\varphi \notin U_n(\pm\varphi_0)$

$$\Re V(\lambda_0^{-1/2}e^{i\varphi}, \lambda) \geq \frac{C \log^2 n}{n}.$$

The lemma is proved. \square

Note that $|v_j| = \lambda_0^{-1/2}$, $j = 1, \dots, 2k$. Since ξ_1, \dots, ξ_{2k} are distinct, the inequality $|\Delta(T)/\Delta(\hat{\xi})| \leq C_1$ and (3.7) yield

$$\left| Z_{2k} \oint_{\omega_0 \setminus (U_{v,+} \cup U_{v,-})} \oint_{\omega_0} \dots \oint_{\omega_0} W_n(v_1, \dots, v_{2k}) \prod_{j=1}^{2k} dv_j \right| \leq C_1 n^{k^2} e^{-C_2 \log^2 n},$$

where

$$\omega_0 = \{z \in \mathbb{C} : |z| = \lambda_0^{-1/2}\}, \quad (3.12)$$

W_n and Z_{2k} are defined in (3.4) and (3.5) respectively, and

$$\begin{aligned} U_{\pm} &= \{\varphi \in (-\pi, \pi] : |\pm \varphi_0 - \varphi| \leq n^{-1/2} \log n\}, \\ U_{v,\pm} &= \{z = \lambda_0^{-1/2} e^{i\varphi} | \varphi \in U_{\pm}\} \end{aligned} \quad (3.13)$$

with φ_0 of (3.6).

Note that we have for $\varphi \in U_{\pm}$ in view of (3.1) and (3.10) as $m, n \rightarrow \infty$

$$V(\lambda_0^{-1/2} e^{i\varphi}, \lambda_0) = V_{\pm} + \left(\frac{1}{v_{\pm}^2} - \frac{c_{m,n}}{(1 - v_{\pm})^2} \right) \lambda_0^{-1} e^{\pm 2i\varphi_0} \frac{(\varphi \mp \varphi_0)^2}{2} + f_{\pm}(\varphi \mp \varphi_0), \quad (3.14)$$

where $f_{\pm}(\varphi \mp \varphi_0) = O((\varphi \mp \varphi_0)^3)$. Shifting $\varphi_j \mp \varphi_0 \rightarrow \varphi_j$ for $\varphi_j \in U_{\pm}$ and using (3.10) we obtain

$$\begin{aligned} \frac{D_{2k}^{-1}(\lambda_0)}{(n\rho(\lambda_0))^{k^2}} F_{2k}(\Lambda_{2k}) &= Z_{2k} \lambda_0^{k(2k-1)/2} \sum_{s=1}^{2k} \sum_{\alpha^s} \int_{(U_n)^{2k}} e^{G_s(\varphi_1, \dots, \varphi_{2k}) + \sum_{j=1}^{2k} d_{\alpha_j}(\varphi_j) \xi_j} \frac{\Delta(V^{\alpha^s})}{\Delta(\widehat{\xi})} \\ \prod_{l=1}^s e^{-\frac{nc_{+}}{2} \varphi_l^2 - (2k-1)i(\varphi_j + \varphi_0)} \prod_{j=s+1}^{2k} e^{-\frac{nc_{-}}{2} \varphi_j^2 - (2k-1)i(\varphi_j - \varphi_0)} \prod_{j=1}^{2k} d\varphi_j &= \sum_{s=1}^{2k} \sum_{\alpha^s} T_{s,\alpha}, \end{aligned} \quad (3.15)$$

where $\alpha^s = \{\alpha_j\}_{j=1}^{2k}$ is a permutation of s pluses and $2k - s$ minuses,

$$d_{\pm}(\varphi_j) = \frac{e^{i(\varphi_j \pm \varphi_0)}}{\sqrt{\lambda_0} \rho(\lambda_0)}, \quad V^s = \text{diag}\{e^{i(\varphi_1 + \varphi_0)}, \dots, e^{i(\varphi_s + \varphi_0)}, e^{i(\varphi_{s+1} - \varphi_0)}, \dots, e^{i(\varphi_{2k} - \varphi_0)}\}, \quad (3.16)$$

$$\begin{aligned} G_s(\varphi_1, \dots, \varphi_{2k}) &= 2c_{m,n} \kappa_4 S_2((I - V^s) \Lambda_0) \prod_{l=1}^s \frac{\lambda_0^{-1/2} e^{i(\varphi_l + \varphi_0)}}{1 - \lambda_0^{-1/2} e^{i(\varphi_l + \varphi_0)}} \prod_{r=s+1}^{2k-s} \frac{\lambda_0^{-1/2} e^{i(\varphi_r - \varphi_0)}}{1 - \lambda_0^{-1/2} e^{i(\varphi_r - \varphi_0)}} \\ &- n \sum_{l=1}^s (f_+(\varphi_l) + V_+(\varphi_l)) - n \sum_{r=s+1}^{2k-s} (f_-(\varphi_r) + V_-(\varphi_r)), \end{aligned}$$

$$\begin{aligned} c_{\pm} &= \left(\frac{1}{v_{\pm}^2} - \frac{c_{m,n}}{(1 - v_{\pm})^2} \right) \lambda_0^{-1} e^{\pm 2i\varphi_0}, \quad U_n = (-n^{-1/2} \log n, n^{-1/2} \log n), \\ V^{\alpha^s} &= \text{diag}\{e^{i(\varphi_1 + \alpha_1 \varphi_0)}, \dots, e^{i(\varphi_{2k} + \alpha_{2k} \varphi_0)}\}. \end{aligned} \quad (3.17)$$

Define

$$\begin{aligned} I_s &:= \int_{\Omega_{n,s}} e^{\frac{1}{\sqrt{n}} \sum_{j=1}^s \xi_j g(\varphi_j)} \prod_{j < l} (\varphi_j - \varphi_l) d\nu_s(\varphi_1, \dots, \varphi_s) \\ &= \int_{\Omega_{n,s}} \det \left\{ e^{\frac{1}{\sqrt{n}} \xi_j g(\varphi_j)} \varphi_j^{l-1} \right\}_{j,l=1}^s d\nu_s(\varphi_1, \dots, \varphi_s) \\ &= \sum_{p_1, \dots, p_s=0}^{\infty} \int_{\Omega_{n,s}} \det \left\{ (n^{-1/2} \xi_j g(\varphi_j))^{p_j} \varphi_j^{l-1} / p_j! \right\}_{j,l=1}^s d\nu_s(\varphi_1, \dots, \varphi_s), \end{aligned} \quad (3.18)$$

where $d\nu_s(\varphi_1, \dots, \varphi_s)$ is a measure on $\Omega_{n,s} := (-\log n, \log n)^s$ which is symmetric in $(\varphi_1, \dots, \varphi_s)$ and $g(\varphi)$ is a function such that $g(\varphi) = C\varphi(1 + o(1))$, $n \rightarrow \infty$. Note that if we take the term of (3.18) such that $p_{s_1} = p_{s_2}$, $s_1 \neq s_2$, then this term is zero since $d\nu_s(\varphi_1, \dots, \varphi_s)$ is symmetric in $(\varphi_1, \dots, \varphi_s)$. Moreover, the order of

$$\det \left\{ (n^{-1/2} \xi_j g(\varphi_j))^{p_j} \varphi_j^{l-1} / p_j! \right\}_{j,l=1}^s$$

is $n^{-(p_1 + \dots + p_s)/2}$ and if $\{p_1, \dots, p_s\} \neq \{0, 1, \dots, s-1\}$ the order is less than $n^{-s(s-1)/2}$. Hence, denoting by $\widetilde{\sum}$ the sum over all permutations $\{p_1, \dots, p_s\}$ of $\{0, 1, \dots, s-1\}$, we obtain

$$\begin{aligned} I_s &= \frac{n^{-s(s-1)/2}}{\prod_{j=0}^{s-1} j!} \widetilde{\sum} \int_{\Omega_{n,s}} \prod_{j=1}^s (\xi_j g(\varphi_j))^{p_j} \Delta(\varphi_1, \dots, \varphi_s) d\nu_s(\varphi_1, \dots, \varphi_s) (1 + o(1)) \\ &= \frac{\Delta(\xi_1, \dots, \xi_s)}{n^{s(s-1)/2} \prod_{j=0}^{s-1} j!} \int_{\Omega_{n,s}} \prod_{j=1}^s g(\varphi_j)^{j-1} \Delta(\varphi_1, \dots, \varphi_s) d\nu_s(\varphi_1, \dots, \varphi_s) (1 + o(1)) \quad (3.19) \\ &= \frac{\Delta(\xi_1, \dots, \xi_s)}{n^{s(s-1)/2} \prod_{j=0}^s j!} \int_{\Omega_{n,s}} \Delta(g(\varphi_1), \dots, g(\varphi_s)) \Delta(\varphi_1, \dots, \varphi_s) d\nu_s(\varphi_1, \dots, \varphi_s) (1 + o(1)), \end{aligned}$$

Since $g(\varphi) = C\varphi(1 + o(1))$, $n \rightarrow \infty$, we get

$$I = \frac{C^{s(s-1)/2} \Delta(\xi_1, \dots, \xi_s)}{n^{s(s-1)/2} \prod_{j=0}^s j!} \int_{\Omega_{n,s}} \Delta^2(\varphi_1, \dots, \varphi_s) d\nu_s(\varphi_1, \dots, \varphi_s) (1 + o(1)). \quad (3.20)$$

Consider T_α of (3.15) with $\alpha_1 = \dots = \alpha_s = +$, $\alpha_{s+1} = \dots = \alpha_{2k} = -$. Since the function $2c_{m,n} \kappa_4 S_2((I - V_\alpha) \Lambda_0) \prod_{l=1}^{2k} \frac{\lambda_0^{-1/2} e^{i(\varphi_l + \alpha_l \varphi_0)}}{1 - \lambda_0^{-1/2} e^{i(\varphi_l + \alpha_l \varphi_0)}}$ is symmetric in $(\varphi_1, \dots, \varphi_s)$ and $(\varphi_{s+1}, \dots, \varphi_{2k})$, changing variables as $\sqrt{n} \varphi_j \rightarrow \varphi_j$ and using formulas (3.18) – (3.20), and formula for the Selberg integral (see, e.g., [14], Chapter 17), we obtain

$$\begin{aligned} T_\alpha &= \frac{C_{0,s}(\hat{\xi})}{n^{(k-s)^2} \prod_{j=0}^s j! \prod_{l=0}^{2k-s} l!} \int_{-\log n}^{\log n} \prod_{j=1}^{2k} d\varphi_j \Delta^2(\varphi_1, \dots, \varphi_s) \prod_{j=1}^s e^{-\frac{c+\varphi_j^2}{2}} \\ &\times \Delta^2(\varphi_{s+1}, \dots, \varphi_{2k}) \prod_{l=s+1}^{2k} e^{-\frac{c-\varphi_l^2}{2}} (1 + o(1)) = \frac{C_{0,s}(\hat{\xi}) (2\pi)^k}{c_+^{s^2/2} c_-^{(2k-s)^2/2} n^{(k-s)^2}} (1 + o(1)), \quad (3.21) \end{aligned}$$

where $C_{0,s}(\hat{\xi})$ is an n -independent constant. This expression is of order $O(1)$ for $s = k$, and it is of order $o(1)$ for $s \neq k$. Hence, only the terms of (3.15) with exactly k of $\{\alpha_j\}_{j=1}^{2k}$ pluses contribute in the limit (1.15). If we take $s = k$ we obtain

$$\begin{aligned} C_{0,k}(\hat{\xi}) &= \lambda_0^{k(2k-1)/2} \left(\frac{e^{2i\varphi_0}}{\lambda_0^{1/2} \rho(\lambda_0)} \right)^{\frac{k(k-1)}{2}} \left(\frac{e^{-2i\varphi_0}}{\lambda_0^{1/2} \rho(\lambda_0)} \right)^{\frac{k(k-1)}{2}} (2i \sin \varphi_0)^{k^2} \\ &\frac{\exp\{i\pi(\xi_1 + \dots + \xi_k - \xi_{k+1} - \dots - \xi_{2k})\}}{\prod_{j=1}^k \prod_{l=k+1}^{2k} (\xi_j - \xi_l)} e^{k(k-1)\kappa_4(c-\lambda_0+1)^2 c^{-1}} \quad (3.22) \\ &= \frac{\lambda_0^{k^2} (2i\pi \rho(\lambda_0))^{k^2}}{(2\pi)^{2k} c^{k/2}} e^{k(k-1)\kappa_4(c-\lambda_0+1)^2 c^{-1}} \frac{e^{i\pi(\xi_1 + \dots + \xi_k - \xi_{k+1} - \dots - \xi_{2k})}}{\prod_{j=1}^k \prod_{l=k+1}^{2k} (\xi_j - \xi_l)}. \end{aligned}$$

Hence, since it is easy to check that

$$c_+c_- = \frac{4\pi^2\lambda_0^2\rho(\lambda_0)^2}{c_{m,n}},$$

we get from (3.21) and (3.22) that T_α of (3.15) with $\alpha_1 = \dots = \alpha_k = +$, $\alpha_{k+1} = \dots = \alpha_{2k} = -$ has the form

$$\frac{i^{k(k+1)} e^{i\pi(\xi_1 + \dots + \xi_k - \xi_{k+1} - \dots - \xi_{2k})}}{(2i\pi)^k \prod_{i,j=1}^k (\xi_i - \xi_{k+j})} c^{k(k-1)/2} e^{k(k-1)\kappa_4(c-\lambda_0+1)^2 c^{-1}} \quad (3.23)$$

In view of the identity

$$\frac{\det \left\{ \frac{\sin(\pi(\xi_j - \xi_{k+l}))}{\pi(\xi_j - \xi_{k+l})} \right\}_{j,l=1}^k}{\Delta(\xi_1, \dots, \xi_k) \Delta(\xi_{k+1}, \dots, \xi_{2k})} = \frac{\det \left\{ \frac{e^{i\pi(\xi_j - \xi_{k+l})} - e^{i\pi(\xi_{k+l} - \xi_j)}}{\xi_j - \xi_{k+l}} \right\}_{j,l=1}^k}{(2i\pi)^k \Delta(\xi_1, \dots, \xi_k) \Delta(\xi_{k+1}, \dots, \xi_{2k})}$$

the determinant in the l.h.s. of (3.23) is a linear combination of $\exp\{i\pi \sum_{j=1}^{2k} \alpha_j \xi_j\}$ over the collection $\{\alpha_j\}_{j=1}^{2k}$, in which m elements are pluses, and the rest are minuses. By the virtue of the following formula (see [18], Problem 7.3)

$$(-1)^{\frac{k(k-1)}{2}} \frac{\prod_{j < l} (a_j - a_l)(b_j - b_l)}{\prod_{j,l=1}^k (a_j - b_l)} = \det \{(a_j - b_l)^{-1}\}_{j,l=1}^m. \quad (3.24)$$

the coefficient of $\exp\{i\pi(\xi_{k+1} + \dots + \xi_{2k} - \xi_1 - \dots - \xi_k)\}$ is

$$\frac{\det \{(\xi_{k+l} - \xi_j)^{-1}\}_{j,l=1}^k}{(2i\pi)^k \Delta(\xi_1, \dots, \xi_k) \Delta(\xi_{k+1}, \dots, \xi_{2k})} = \frac{(-1)^{\frac{k(k-1)}{2}}}{(-1)^{k^2} (2i\pi)^k \prod_{i,j=1}^k (\xi_i - \xi_{k+j})}.$$

Other coefficients can be computed analogously. Thus, restricting the sum in (3.15) to that over the collection $\{\alpha_j\}_{j=1}^{2k}$, in which exactly k elements are pluses, and k are minuses, and using (3.23), we obtain Theorem 1 after a certain algebra.

4 Asymptotic analysis at the edge of the spectrum.

Let now $\lambda_0 = \lambda_+$ (for $\lambda_0 = \lambda_-$ the proof is similar) and $\lambda_j = \lambda_+ + \xi_j / (n\gamma_+)^{2/3}$, $j = 1, \dots, 2k$, where λ_+ and γ_+ are defined in (1.6) and (1.5), and $\xi_1, \dots, \xi_{2k} \in [-M, M] \subset \mathbb{R}$.

According to (2.3) we have

$$\frac{D_{2k}^{-1}(\lambda_+)}{(n\gamma_+)^{2k^2/3}} F_{2k}(\Lambda_{2k}) = \widetilde{Z}_{2k} \oint_{\omega_0} \widetilde{W}_n(v_1, \dots, v_{2k}) \prod_{j=1}^{2k} dv_j (1 + o(1)), \quad (4.1)$$

where D_{2k} is defined in (1.14),

$$\widetilde{W}_n(v_1, \dots, v_{2k}) = \exp \left\{ -n \sum_{l=1}^{2k} V^{(+)}(v_l) + \sum_{l=1}^{2k} \frac{n^{1/3} \xi_l}{\gamma^{2/3}} v_l + n(c_{m,n} - c) \sum_{l=1}^{2k} \log(1 - v_l) \right\} \quad (4.2)$$

$$\times \frac{\Delta(V)}{\Delta(\widehat{\xi})} \exp \left\{ 2c_{m,n} \kappa_4 S_2((I - V)\Lambda_0) \prod_{l=1}^{2k} \frac{v_l}{1 - v_l} \right\} \prod_{j=1}^{2k} \frac{1}{v_j^{2k}},$$

$$V^{(+)}(v) = -\lambda_0 v - c \log(1 - v) + \log v - S_+, \quad (4.3)$$

$$S_+ = -1 - \sqrt{c} - c \log(1 - (1 + \sqrt{c})^{-1}) - \log(1 + \sqrt{c}), \quad (4.4)$$

and

$$\widetilde{Z}_{2k} = L_{2k} e^{-n^{1/3} \alpha(\lambda_+) \sum_{j=1}^{2k} \xi_j + 2k(nc-m) \log(1 - \lambda_+^{-1/2})}, \quad (4.5)$$

$$L_{2k} = \frac{n^{k(2k+1)/3} \gamma^{2k(k-1)/3} e^{-2k\kappa_4 - 2k(nc-m) \log(1 - \lambda_+^{-1/2})}}{2^{2k} \pi^{2k} c^{k/2}}. \quad (4.6)$$

We need the following lemma

Lemma 4. *The function $\Re V^{(+)}(v)$ for $v = \lambda_+^{-1/2} e^{i\varphi}$, $\varphi \in (-\pi, \pi]$ attains its minimum at*

$$v_0 := \lambda_+^{-1/2} = (1 + \sqrt{c})^{-1}. \quad (4.7)$$

Moreover, if $v \in \omega_0 = \{v \in \mathbb{C} : v = \lambda_+^{-1/2} e^{i\varphi}, \varphi \in (-\pi, \pi]\}$, $|v - v_0| \geq \delta$, where δ is small enough, then we have for sufficiently big n

$$\Re V^{(+)}(v) \geq C\delta^4. \quad (4.8)$$

Proof. Similarly to (3.8) – (3.9) we have

$$\frac{d}{d\varphi} \Re V^{(+)}(v_0) = \frac{d^2}{d\varphi^2} \Re V^{(+)}(v_0) = \frac{d^3}{d\varphi^3} \Re V^{(+)}(v_0) = 0, \quad (4.9)$$

$$\frac{d^4}{d\varphi^4} \Re V^{(+)}(v_0) = 6. \quad (4.10)$$

Hence, $\varphi = 0$ is a minimum point of the function $\Re V^{(+)}(\lambda_+^{-1/2} e^{i\varphi})$, and $\Re V^{(+)}(\lambda_+^{-1/2} e^{i\varphi})$ is monotone increasing function for $\varphi \in [0, \pi)$ and monotone decreasing function for $\varphi \in (-\pi, 0]$.

Expanding $\Re V^{(+)}(\lambda_+^{-1/2} e^{i\varphi})$ into the Taylor series we obtain for $|\varphi| \leq \delta$ similarly to (3.11):

$$\Re V^{(+)}(\lambda_+^{-1/2} e^{i\varphi}) = \varphi^4/4 + O(\varphi^5). \quad (4.11)$$

This and monotonicity of $\Re V^{(+)}(\lambda_+^{-1/2} e^{i\varphi})$ for $\varphi \neq 0$ imply for $\varphi \in (-\pi, \pi]$, $|\varphi| \geq \delta$

$$\Re V^{(+)}(\lambda_+^{-1/2} e^{i\varphi}) \geq C\delta^4.$$

Since $|v - v_0| = 2\lambda_+^{-1/2} |\sin(\varphi/2)| \leq \lambda_+^{-1/2} |\varphi|$, we get (4.11). \square

Note that $|v_j| = \lambda_+^{-1/2}$, $j = 1, \dots, 2k$ and according to (1.16) $c_{m,n} - c = o(n^{-2/3})$, $m, n \rightarrow \infty$. Since ξ_1, \dots, ξ_{2k} are distinct, the inequality $|\Delta(T)/\Delta(\widehat{\xi})| \leq C_1$ and (4.8) yield

$$\left| \widetilde{Z}_{2k} \int_{\omega_0 \setminus U_\delta(v_0)} \oint_{\omega_0} \dots \oint_{\omega_0} \widetilde{W}_n(v_1, \dots, v_{2k}) \prod_{j=1}^{2k} dv_j \right| \leq C_1 n^{k(2k+1)/3} e^{-C_2 n(1+o(1)) + C_3 n^{1/3}}, \quad m, n \rightarrow \infty,$$

where

$$U_\delta(v_0) = \{v \in \omega_0 : |v - v_0| \leq \delta\}.$$

Hence,

$$\frac{D_{2k}^{-1}(\lambda_+)}{(n\gamma_+)^{2k^2/3}} F_{2k}(\Lambda_{2k}) = \widetilde{Z}_{2k} \oint_{U_\delta(v_0)} \widetilde{W}_n(v_1, \dots, v_{2k}) \prod_{j=1}^{2k} dv_j (1 + o(1)) + O(e^{-Cn}). \quad (4.12)$$

Since

$$\frac{d}{dv} V^{(+)}(v_0) = \frac{d^2}{dv^2} V(v_0) = 0,$$

we have for $|v - v_0| \leq \delta$

$$V^{(+)}(v) = \gamma^{-2}(v - v_0)^3/3 + O((v - v_0)^4), \quad |v - v_0| \rightarrow 0. \quad (4.13)$$

Thus, we can write for v satisfying $|v - v_0| \leq \delta$

$$V^{(+)}(v) = \gamma^{-2} \chi^3(v)/3, \quad (4.14)$$

where $\chi(v)$ is analytic in the δ -neighborhood of v_0 with the analytic inverse $z(\varphi)$ (we choose $\chi(v)$ such that $\chi(v) \in \mathbb{R}$ for $v \in \mathbb{R}$).

Changing variables to $v_j = z(\varphi_j)$, $j = 1, \dots, 2k$, we rewrite (4.12) as

$$\begin{aligned} \frac{D_{2k}^{-1}(\lambda_+)}{(n\gamma_+)^{2k^2/3}} F_{2k}(\Lambda_{2k}) &= L_{2k} \oint_{\widetilde{U}_{\delta,\varphi}} e^{-n\gamma^{-2} \sum_{l=1}^{2k} \varphi_l^3/3 + \sum_{l=1}^{2k} \frac{n^{1/3} \xi_l}{\gamma^{2/3}} (z(\varphi_l) - v_0) + n(c_{m,n} - c) \sum_{l=1}^{2k} \log \frac{1 - z(\varphi_l)}{1 - v_0}} \\ &\times e^{2c_{m,n} \kappa_4 S_2((I-Z)\Lambda_0) \prod_{l=1}^{2k} \frac{z(\varphi_l)}{1 - z(\varphi_l)} \frac{\Delta(Z)}{\Delta(\widehat{\xi})} \prod_{j=1}^{2k} \frac{z'(\varphi_j)}{z^{2k}(\varphi_j)} dv_j (1 + o(1)) + O(e^{-Cn})} \\ &=: L_{2k} \int_{\widetilde{U}_{\delta,\varphi}} \widehat{W}(\varphi_1, \dots, \varphi_{2k}) \prod_{j=1}^{2k} dv_j (1 + o(1)) + O(e^{-Cn}), \end{aligned} \quad (4.15)$$

where L_{2k} is defined in (4.5),

$$Z = \text{diag} \{z(\varphi_1), \dots, z(\varphi_{2k})\}, \quad (4.16)$$

$$\widetilde{U}_{\delta,\varphi} = \{\varphi \in \mathbb{C} : z(\varphi) \in U_\delta(v_0)\}. \quad (4.17)$$

Moreover, we have from (4.14)

$$\chi(v_0) = 0, \quad \frac{d}{dz}\chi(v_0) = 1, \quad (4.18)$$

hence

$$0 < C_1 < |\chi'(v)| < C_2, \quad |v - v_0| \leq \delta. \quad (4.19)$$

If $\tilde{\sigma} = \{z \in \mathbb{C} : |z - z_{0,n}^*| \leq \delta\}$, then $\chi(\partial\tilde{\sigma})$ is a closed curve encircling $\varphi = 0$ and lying between the circles $\sigma_1 = \{\varphi \in \mathbb{C} : |\varphi| = C_1\delta\}$ and $\sigma_2 = \{\varphi \in \mathbb{C} : |\varphi| = C_2\delta\}$ for $0 < C_1 < C_2$. We have from (4.18)

$$z(0) = v_0, \quad z'(0) = 1, \quad 0 < C_1 < |z'(\varphi)| < C_2, \quad \varphi \in \chi(\tilde{\sigma}). \quad (4.20)$$

According to Lemma 4, $\Re V^{(+)}(v) \geq 0$ for $v \in U_\delta(v_0)$ and we get $\Re \varphi_j^3 \geq 0$ for $\varphi_j \in \tilde{U}_{\delta,\varphi}$, i.e.,

$$\cos(3 \arg \varphi_j) \geq 0, \quad \varphi_j \in \tilde{U}_{\delta,\varphi},$$

where $\tilde{U}_{\delta,\varphi}$ is defined in (4.17). Hence, $\tilde{U}_{\delta,\varphi}$ can be located only in the sectors

$$-\pi/6 \leq \arg \varphi \leq \pi/6, \quad \pi/2 \leq \arg \varphi \leq 5\pi/6, \quad 7\pi/6 \leq \arg \varphi \leq 3\pi/2.$$

Besides, χ is conformal in $\tilde{\sigma}$ (see (4.19)), hence angle-preserving. Taking into account that $\chi(v) \in \mathbb{R}$ for $v \in \mathbb{R}$, the angle between ω_0 and the real axis at the point v_0 is $\pi/2$, and that $\tilde{U}_{\delta,\varphi}$ is a continuous curve, we obtain that $\tilde{U}_{\delta,\varphi}$ can be located only in the sectors

$$\pi/2 \leq \arg \varphi \leq 5\pi/6, \quad 7\pi/6 \leq \arg \varphi \leq 3\pi/2. \quad (4.21)$$

Note that we can take any curve $\tilde{U}(\varphi)$ instead of $\tilde{U}_{\delta,\varphi}$ provided that $\tilde{U}(\varphi)$ and $\omega_0 \setminus U_\delta(v_0)$ are "glued", i.e., the union of $z(\tilde{U}(\varphi))$ and $\omega_0 \setminus U_\delta(v_0)$ form a closed contour encircling 0. Let us take

$$\begin{aligned} \tilde{U}(\varphi) = \{ \varphi \in \mathbb{C} : \arg \varphi = 2\pi/3, \varphi \in \chi(\tilde{\sigma}) \} \\ \cup \{ \varphi \in \mathbb{C} : \arg \varphi = 4\pi/3, \varphi \in \chi(\tilde{\sigma}) \} \cup U_{1,\delta} \cup U_{2,\delta}, \end{aligned}$$

where $\tilde{\sigma} = \{v \in \mathbb{C} : |v - v_0| \leq \delta\}$, $U_{1,\delta}$ is a curve along $\chi(\partial\tilde{\sigma})$ from the point of intersection of the ray $\arg \varphi = 2\pi/3$ and $\chi(\partial\tilde{\sigma})$ to the point $\varphi_{1,\delta}$ of intersection of $\tilde{U}_{\delta,\varphi}$ and $\chi(\partial\tilde{\sigma})$ ($\pi/2 < \arg \varphi_{1,\delta} < 5\pi/6$), and $U_{2,\delta}$ is a curve along $\chi(\partial\tilde{\sigma})$ from the point of intersection of the ray $\arg \varphi = 4\pi/3$ and $\chi(\partial\tilde{\sigma})$ to the point $\varphi_{2,\delta}$ of intersection of $\tilde{U}_{\delta,\varphi}$ and $\chi(\partial\tilde{\sigma})$ ($7\pi/6 < \arg \varphi_{2,\delta} < 3\pi/2$). According to Lemma 4 and (4.14), $\Re \varphi_{1,\delta}^3 = r^3 \cos 3\varphi_0 > C > 0$, where $r = |\varphi_{1,\delta}|$, $\varphi_0 = \arg \varphi_{1,\delta}$. Since $0 < C_1 < r < C_2$, we have

$$\cos 3\varphi_0 \geq C/C_2^3 > 0.$$

Moreover, it is easy to see that $\cos(3 \arg \varphi_1) > \cos 3\varphi_0$ along $U_{1,\delta}$ (since $\cos 3x$ is monotone increasing for $x \in [\pi/2, 2\pi/3]$ and monotone decreasing for $x \in [2\pi/3, 5\pi/6]$). This and $|\varphi_j| > C_1$ imply for $\varphi_j \in L_{1,\delta}$

$$\Re \left(\frac{\gamma^{-2} \varphi_j^3}{3} \right) > C > 0, \quad \varphi_1 \in U_{1,\delta}. \quad (4.22)$$

Also we have from (4.20)

$$|z(\varphi_j) - v_0| \leq C_2 |\varphi_j| < C, \quad \varphi_j \in \chi(\tilde{\sigma}).$$

This, (4.22), $c_{m,n} - c = o(n^{-2/3})$, $m, n \rightarrow \infty$, and (4.20) yield

$$\left| \widehat{W}(\varphi_1, \dots, \varphi_{2k}) \right| \leq e^{-Cn+o(n)}, \quad \varphi_1 \in U_{1,\delta}, \quad \varphi_j \in \tilde{U}(\varphi_j), \quad j > 1. \quad (4.23)$$

Hence, the integral over $U_{1,\delta}$ does not contribute to the l.h.s. of (4.15). The same statement we can prove for $U_{2,\delta}$. Thus, we have shown that integral over $\tilde{U}_{\delta,\varphi}$ in (4.15) can be replaced to the integral over the contour

$$\tilde{l} = \{\varphi \in \mathbb{C} : \arg \varphi = 2\pi/3, \varphi \in \chi(\tilde{\sigma})\} \cup \{\varphi \in \mathbb{C} : \arg \varphi = 4\pi/3, \varphi \in \chi(\tilde{\sigma})\}. \quad (4.24)$$

According to the choice of \tilde{l} , we have

$$\Re \varphi_j^3 = r_j^3, \quad \varphi_j \in \tilde{l}, \quad (4.25)$$

where $r_j = |\varphi_j|$.

Set now

$$\sigma_n = \{\varphi \in \mathbb{C} : |\varphi| \leq \varepsilon_n^{-1/2} n^{-1/3}\},$$

where ε_n is defined in (1.16). Note that we assume that $\varepsilon_n^{-1/2} n^{-1/3} \rightarrow 0$, $n \rightarrow \infty$. In other case we can take $\sigma_n = \{\varphi \in \mathbb{C} : |\varphi| \leq \log n / n^{1/3}\}$ and the proof will be similarly.

It is easy to see that $\sigma_n \subset \chi(\tilde{\sigma})$ for sufficiently big n . Besides, we have from (4.20) for $\varphi \in \sigma_n$

$$\begin{aligned} z(\varphi) &= v_0 + \varphi + O(\varepsilon_n^{-1} n^{-2/3}), \quad n \rightarrow \infty, \\ z'(\varphi) &= 1 + O(\varepsilon_n^{-1/2} n^{-1/3}), \quad n \rightarrow \infty. \end{aligned} \quad (4.26)$$

Taking into account (1.16), (4.26), (4.25), and

$$\left| \log \frac{1 - z(\varphi)}{1 - v_0} \right| \leq \left| \frac{v_0 - z(\varphi)}{1 - v_0} \right|,$$

we obtain for $\varphi_1 \in \tilde{l} \setminus \sigma_n$, $\varphi_j \in \tilde{l}$, $j = 2, \dots, 2k$

$$\left| \widehat{W}(\varphi_1, \dots, \varphi_{2k}) \right| \leq C_1 e^{-C_2 n r_1^3 + C_3 n^{1/3} r_1} \quad (4.27)$$

where $r_1 = |\varphi_1| \geq \varepsilon_n^{-1/2} n^{-1/3}$. Since $n^{1/3} r_1 \geq \varepsilon_n^{-1/2}$ for $\varphi_1 \in \tilde{l} \setminus \sigma_n$, the integral over $\tilde{l} \setminus \sigma_n$ is $O(e^{-C\varepsilon_n^{-3/2}})$ as $n \rightarrow \infty$. Hence,

$$\frac{D_{2k}^{-1}(\lambda_+)}{(n\gamma_+)^{2k^2/3}} F_{2k}(\Lambda_{2k}) = L_{2k}(1 + o(1)) \int_{\tilde{l} \cap \sigma_n} \widehat{W}(\varphi_1, \dots, \varphi_{2k}) \prod_{j=1}^{2k} d\varphi_j + O(e^{-C\varepsilon_n^{-3/2}}), \quad (4.28)$$

where L_{2k} and $\widehat{W}(\varphi_1, \dots, \varphi_{2k})$ are defined in (4.5) and (4.15). This, (2.4), and (4.26) imply

$$\begin{aligned} \frac{D_{2k}^{-1}}{(n\gamma_+)^{2k^2/3}} F_{2k}(\Lambda_{2k}) &= \frac{L_{2k} e^{2k(2k-1)\kappa_4}}{v_0^{4k^2}} \int_{\tilde{l} \cap \sigma_n} e^{-n\gamma^{-2} \sum_{l=1}^{2k} \varphi_l^3 / 3 + \sum_{l=1}^{2k} \frac{n^{1/3} \xi_l}{\gamma^{2/3}} \varphi_l} \frac{\Delta(\Phi)}{\Delta(\hat{\xi})} \\ &\quad \times (1 + \delta(\varphi_1, \dots, \varphi_{2k})) \prod_{j=1}^{2k} d\varphi_j + O(e^{-C\varepsilon_n^{-3/2}}), \end{aligned} \quad (4.29)$$

where $\delta(\varphi_1, \dots, \varphi_{2k})$ collects the reminder terms which appear when we replace $z(\varphi_j) \rightarrow v_0 + \varphi_j + O(\varphi_j^2)$, $z'(\varphi_j) \rightarrow 1 + O(\varphi_j)$ and $\log \frac{1-z(\varphi_j)}{1-v_0} \rightarrow \frac{\varphi_j}{1-v_0} + O(\varphi_j^2)$, $j = 1, \dots, 2k$. Hence

$$|\delta(\varphi_1, \dots, \varphi_{2k})| \leq C(|\varphi_1| + \dots + |\varphi_{2k}|).$$

Changing variables in (4.29) as $\gamma^{-2/3}n^{1/3}\varphi_j \rightarrow i\varphi_j$ we obtain in new variables

$$|\tilde{\delta}(\varphi_1, \dots, \varphi_{2k})| = |\delta(i\gamma^{2/3}n^{-1/3}\varphi_1, \dots, i\gamma^{2/3}n^{-1/3}\varphi_{2k})| \leq Cn^{-1/3}(|\varphi_1| + \dots + |\varphi_{2k}|).$$

Therefore, using (1.9), (4.5), and (4.7), we obtain

$$\begin{aligned} \frac{D_{2k}^{-1}(\lambda_+)}{(n\gamma_+)^{2k^2/3}} F_{2k}(\Lambda_{2k}) &= \frac{L_{2k}e^{2k(2k-1)\kappa_4}(1+o(n^{-1/3}))}{v_0^{4k^2}(-i\gamma^{-2/3}n^{1/3})^{k(2k+1)}} \\ &\times \int_S e^{i\sum_{l=1}^{2k}\varphi_l^3/3 + \sum_{l=1}^{2k}i\xi_l\varphi_l} \frac{\Delta(\Phi)}{\Delta(\widehat{\xi})} \prod_{j=1}^{2k} d\varphi_j + O(e^{-C\varepsilon_n^{-3/2}}) \\ &= e^{4k(k-1)\kappa_4}(-1)^{k^2}c^{k(k-1)/2}(1+o(n^{-1/3})) \\ &\times \frac{i^k}{(2\pi)^{2k}} \int_S e^{i\sum_{l=1}^{2k}\varphi_l^3/3 + \sum_{l=1}^{2k}i\xi_l\varphi_l} \frac{\Delta(\Phi)}{\Delta(\widehat{\xi})} \prod_{j=1}^{2k} d\varphi_j + O(e^{-C\varepsilon_n^{-3/2}}), \end{aligned} \quad (4.30)$$

where S is defined in (1.12).

Consider

$$\begin{aligned} K(\widehat{\xi}) &:= \frac{i^k}{(2\pi)^{2k}} \int_S e^{i\sum_{l=1}^{2k}\varphi_l^3/3 + \sum_{l=1}^{2k}i\xi_l\varphi_l} \Delta(\Phi) \prod_{j=1}^{2k} d\varphi_j \\ &= \frac{i^k}{(2\pi)^{2k}} \int \det \left\{ \varphi_l^{j-1} e^{i\varphi_l^3/3 + i\xi_l\varphi_l} \right\}_{j,l=1}^{2k} \prod_{j=1}^{2k} d\varphi_j. \end{aligned} \quad (4.31)$$

Integrating by parts, we have for $j \geq 3$

$$\begin{aligned} i \int_S \varphi_l^{j-1} e^{i\varphi_l^3/3 + i\xi_l\varphi_l} d\varphi_l &= \int_S \varphi_l^{j-3} e^{i\xi_l\varphi_l} \frac{d}{d\varphi_l} e^{i\varphi_l^3/3} d\varphi_l \\ &= - \int_S ((j-3)\varphi_l^{j-4} + i\xi_l\varphi_l^{j-3}) e^{i\varphi_l^3/3 + i\xi_l\varphi_l} d\varphi_l. \end{aligned}$$

Applying this identity to each line, starting from third, we observe that the first term in the r.h.s. gives zero contribution. Repeating this procedure and replacing and rearranging the lines, we obtain from (4.31)

$$K(\widehat{\xi}) = \frac{i^k}{(2\pi)^{2k}} \int (-1)^{k(k-1)/2} \det \left\{ \varphi_l^{q_j} \xi_l^{r_j} e^{i\varphi_l^3/3 + i\xi_l\varphi_l} \right\}_{j,l=1}^{2k} \prod_{j=1}^{2k} d\varphi_j, \quad (4.32)$$

where $j = q_j k + r_j$, $q_j = 0, 1$, $r_j = 0, 1, \dots, k-1$. Thus,

$$\begin{aligned} \frac{K(\widehat{\xi})}{\Delta(\widehat{\xi})} &= \frac{i^k}{(2\pi)^{2k}} \int_S \sum_{j_1 < \dots < j_k} \frac{\Delta(\xi_{j_1}, \dots, \xi_{j_k}) \overline{\Delta}(\xi_{j_1}, \dots, \xi_{j_k}) \prod_{s=1}^k \varphi_{j_s}}{(-1)^{j_1 + \dots + j_k} \Delta(\xi_1, \dots, \xi_{2k})} e^{i \sum_{l=1}^{2k} \varphi_l^3/3 + \sum_{l=1}^{2k} i \xi_l \varphi_l} \prod_{j=1}^{2k} d\varphi_j \\ &= \frac{i^k}{(2\pi)^{2k}} \int_S \sum_{j_1 < \dots < j_k} \frac{(-1)^{k(k+1)/2} \prod_{s=1}^k \varphi_{j_s}}{\prod_{s=1}^k \prod_{t \neq j_1, \dots, j_k} (\varphi_{j_s} - \varphi_t) \text{sign}(t - j_s)} e^{i \sum_{l=1}^{2k} \varphi_l^3/3 + \sum_{l=1}^{2k} i \xi_l \varphi_l} \prod_{j=1}^{2k} d\varphi_j, \end{aligned} \quad (4.33)$$

where the sum is over all collections $1 \leq j_1 < \dots < j_k \leq 2k$ and $\overline{\Delta}(\xi_{j_1}, \dots, \xi_{j_k})$ is the Vandermonde determinant of $\{\xi_j\}$ with $j \neq j_1, \dots, j_k$. Consider

$$\frac{(-1)^{k^2}}{(2\pi)^{2k}} \int_S \frac{\det \left\{ \frac{i\varphi_j - i\varphi_{l+k}}{\xi_j - \xi_{l+k}} \right\}_{j,l=1}^k}{\Delta(\xi_1, \dots, \xi_k) \Delta(\xi_{k+1}, \dots, \xi_{2k})} e^{i \sum_{l=1}^{2k} \varphi_l^3/3 + \sum_{l=1}^{2k} i \xi_l \varphi_l} \prod_{j=1}^{2k} d\varphi_j. \quad (4.34)$$

According to the identity (3.24) the coefficient of $\prod_{s=1}^k \varphi_{j_s}$ in (4.34) is

$$\frac{i^k}{(2\pi)^{2k}} \cdot \frac{(-1)^{k(k+1)/2}}{\prod_{s=1}^k \prod_{t \neq j_1, \dots, j_k} (\varphi_{j_s} - \varphi_t) \text{sign}(t - j_s)}.$$

Thus, $K(\widehat{\xi})/\Delta(\widehat{\xi})$ is equal to (4.34), and (4.30) yield the assertion of Theorem 2.

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